Homework 2

Due: Wednesday, February 19

MAT 308, Spring 2025

Problem 1. Prove that there are infinitely many solutions $y\colon \mathbb{R} \to \mathbb{R}$ to the initial value problem

$$y' = \begin{cases} \sqrt{y}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$
$$y(0) = 0$$

Problem 2. For each of the following initial value problems, use Euler's method with a step size of $\Delta x = 0.1$ to give an approximation of $y(1)$, where y is a solution to the equation. Give your approximation in decimal form, up to 3 decimal places. (You will probably want to use a calculator or computer program.)

(a) $y' = x^4 - y^3;\ y(0) = 1$

(b) $y' = e^{\sin(x+y)};\ y(0) = 0$

(c) $y' = (1 + y^2)^{3/2};\ y(0) = 0$

Problem 3. Find a solution for each of the following initial value problems, and prove that the solution you found is the unique solution.

(a) $y' + xy = x;\ y(0) = 1$ where $y\colon \mathbb{R} \to \mathbb{R}$

(b) $xy' + y = x^3;\ y(1) = 1$ where $y\colon (0, \infty) \to \mathbb{R}$

(c) $y' = y + 1;\ y(0) = -1$ where $y\colon \mathbb{R} \to \mathbb{R}$

Problem 4. Suppose V is a set and $+\colon V \times V \to V$ is a function which is associative and commutative, in the sense that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ and $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(a) Suppose there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for every $\mathbf{v} \in V$. Prove that there is then a *unique* such element.

(b) Let $\mathbf{v} \in V$ and suppose that there is an element $(-\mathbf{v})$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. Prove that there is then a *unique* such element.

Now let W be a vector space.

(c) Prove that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W$, if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

(d) Prove that $0 \cdot \mathbf{w} = \mathbf{0}$ for any $\mathbf{w} \in W$. *Hint: First prove that* $0 \cdot \mathbf{w} + 0 \cdot \mathbf{w} = 0 \cdot \mathbf{w}$.

Problem 5. Let $f\colon \mathbb{C}^m \to \mathbb{C}^n$ be a linear map. Prove that there are elements $a_{ij} \in \mathbb{C}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$f \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + \cdots + a_{1m}v_m \\ \vdots \\ a_{n1}v_1 + \cdots + a_{nm}v_m \end{bmatrix}$$

for all $\mathbf{v} \in V$.

Problem 6. Let $f\colon V \to W$ be a linear map between two vector spaces.

(a) Prove that $f(\mathbf{0}) = \mathbf{0}$. *Hint: Problem 4 (d).*

(b) Prove that f is injective if and only if $\ker(f) = \{\mathbf{0}\} \subset V$.