Homework 3 (Updated Mar. 5)

Due: Wednesday, March 5

MAT 308, Spring 2025

Problem 1. Let K be either $\mathbb{R}$ or $\mathbb{C}$, and let S be any set. Recall that we give the set K^S of functions $S \to K$ the structure of a K-vector space by defining $f + g\colon S \to K$ by $(f+g)(s) = f(s) + g(s)$ and defining $(a \cdot f)\colon S \to K$ by $(a \cdot f)(s) = a \cdot (f(s))$ for $f, g \in K^S$ and $a \in K$.

Verify that $(K^S, +, \cdot)$ satisfies all of the axioms of a K-vector space.

Problem 2. Prove that the vector space $\mathbb{R}[x]$ of polynomial functions on $\mathbb{R}$ is infinite-dimensional.

Problem 3. Let K be either $\mathbb{R}$ or $\mathbb{C}$. Prove that the K-vector space $K^{m\times n}$ of $(m \times n)$-matrices has dimension $m \cdot n$.

Problem 4. Let K be either $\mathbb{R}$ or $\mathbb{C}$. In this and the next problem, we will write K^N for the vector space $K^{N\times 1}$ of column vectors over K.

Let $f\colon K^l \to K^m$ and $g\colon K^m \to K^n$ be linear maps given by multiplication by $A \in K^{m\times l}$ and $B \in K^{n\times m}$, respectively, i.e., $f(\mathbf{u}) = A \cdot \mathbf{u}$ and $g(\mathbf{v}) = B \cdot \mathbf{v}$ for all $\mathbf{u} \in K^l$ and $\mathbf{v} \in K^m$.

Prove that $g \circ f\colon K^l \to K^n$ is given by multiplication by $B \cdot A \in K^{n\times l}$ (the matrix product). *Hint: You may use that matrix multiplication is associative.*

Problem 5. Suppose that a symmetric matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ satisfies $a > 0$ and $ad - b^2 > 0$. Prove that the bilinear form $\beta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $\beta(\mathbf{u}, \mathbf{v}) = \mathbf{u}^\top \cdot A \cdot \mathbf{v}$ is positive definite (and hence an inner product, since we already know it to be bilinear and symmetric).

Problem 6. Consider the vector space $V = \mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers; an element of V is a sequence $(a_n)_{n\geq 0}$, and addition and scalar multiplication are given by $(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0}$ and $r \cdot (a_n)_{n\geq 0} = (r \cdot a_n)_{n\geq 0}$ for $r \in \mathbb{R}$.

(a) Find a linear map $V \to V$ which is injective but not surjective.

(b) Find a linear map $V \to V$ which is surjective but not injective.

(c) Let $W \subset V$ be the subspace consisting of all $(a_n)_{n\geq 0}$ that *vanish for almost all* n, in the sense that there is some $N \in \mathbb{N}$ such that $a_n = 0$ for $n \geq N$. Prove that W is isomorphic to the space $\mathbb{R}[x]$ of polynomial functions on $\mathbb{R}$.

(d) With $W \subset V$ as in part (c), define an inner product on W by the formula $\langle (a_n)_{n\geq 0}, (b_n)_{n\geq 0}\rangle = \sum_{n\geq 0} a_n \cdot b_n$ (this is a finite sum by the assumption that a_n and b_n vanish for almost all n).

Prove that the map $\rho\colon W \to W^*$ defined by $\rho(\mathbf{w})(\mathbf{u}) = \langle \mathbf{w}, \mathbf{u}\rangle$ is *not* surjective.

Hint: Each $(a_n)_{n\geq 0} \in W$ *has a well-defined sum* $\sum_{n\geq 0} a_n \in \mathbb{R}$.

Update Mar. 3: $(S, +, \cdot)$ changed to $(K^S, +, \cdot)$ in Problem 1.

Update Mar. 5: $f(\mathbf{v})$ changed to $g(\mathbf{v})$ in Problem 4.