Notes for MAT 308, Spring 2025

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1 More on complex exponentials

- Last time, we defined the exponential function as $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
- A consequence of this is that we can immediately make sense of e^z for *complex* values $z \in \mathbb{C}$, simply by plugging z into the power series.
  - In particular, for $x \in \mathbb{R}$, we find that

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n!} + i\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

  - We recognize the two terms appearing as the Taylor series for cos and sin.
  - As with the exponential function, let us *define* cos and sin to be given by these two power series.
- It follows immediately from this definition that $\sin' = \cos$ and $\cos' = -\sin$.
  - Soon, we will see that there are in fact *unique* functions satisfying the initial value problems $y'' = -y; y(0) = 0; y'(0) = 1$ and $y'' = -y; y(0) = 1; y'(0) = 0$, respectively. Hence, as with the exponential function, we get a nice condition characterizing cos and sin uniquely, and the above series prove the existence of functions satisfying these conditions.
  - It is also not too hard to show from these definitions that $\cos^2 + \sin^2 = 1$ and that they agree with the geometric definition of the trigonometric functions, i.e., that $\mathbf{p} = (\cos(\theta), \sin(\theta)) \in \mathbb{R}^2$ is the point on the unit circle such that the arc-length from $(1, 0)$ is $\mathbf{p}$, measured counter-clockwise, is θ.
- This way of defining e^x, cos, and sin immediately gives rise to the famous *Euler's formula*:

$$e^{ix} = \cos x + i \sin x.$$

  - This gives a concise way to express points in the plane using polar coordinates: the point with radius r and angle θ is $re^{i\theta}$.
    * As usual with radial coordinates, the angle is not uniquely determined: we always have $re^{i\theta} = re^{i(\theta+2\pi)}$ (and conversely, if $r_1e^{i\theta_1} = r_2e^{i\theta_2}$, then $r_1 = r_2$ and $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$ – the one exception being that $0e^{i\theta} = 0$ for *any* θ)

1.1 Some properties of the complex exponential

- A variant (using a bit of complex analysis) of the argument given above to deduce the exponential law $e^{a+b} = e^a \cdot e^b$ for $a, b \in \mathbb{R}$ proves that this holds as well for $a, b \in \mathbb{C}$.

  - From this, we can deduce the addition laws for sin and cos:

$$\begin{aligned} &\cos(a+b) + i\sin(a+b) = e^{i(a+b)} = e^{ia}e^{ib} \\ &= (\cos a + i\sin a)(\cos b + i\sin b) \\ &= (\cos a\cos b - \sin a\sin b) + i(\sin a\cos b + \cos a\sin b) \end{aligned}$$

    hence by comparing real and imaginary parts, we get $\cos(a+b) = \cos a\cos b - \sin a\sin b$ and $\sin(a+b) = \sin a\cos b + \cos a\sin b$.
  - This also makes complex numbers easy to multiply when written in polar coordinates: $(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = (r_1r_2)e^{i(\theta_1+\theta_2)}$.
  - In particular, this allows us to easily find square roots (and more generally n-th roots): if $z = re^{i\theta}$, then its square roots – i.e., the numbers $w \in \mathbb{C}$ such that $w^2 = z$ – are just $w = \pm\sqrt{r}e^{i\theta/2}$.
    - Indeed, we see that these *are* square roots, and given any other square root w, we have $w^2 - z = (w - \sqrt{r}e^{i\theta/2})(w + \sqrt{r}e^{i\theta/2})$, and hence $w = \pm\sqrt{r}e^{i\theta/2}$.
    - (Regarding the ambiguity of θ: had we written $z = re^{i\theta+2pi}$, we would have gotten the same square roots $w = \pm\sqrt{r}e^{i\theta/2+\pi} = \mp\sqrt{r}e^{i\theta/2}$.
  - In particular, if z is a negative real number, then $z = re^{i\pi}$, and we have the familiar imaginary square root $\sqrt{z} = \pm\sqrt{r}e^{i\pi/2} = \pm i\sqrt{r}$.

- Next, recall that the derivative of a function $f\colon \mathbb{R} \to \mathbb{C} = \mathbb{R}^2$ is defined component-wise: if $f(x) = u(x) + iv(x)$, then $f'(x) = u'(x) + iv'(x)$.

  - It follows that
$$\frac{\mathrm{d}}{\mathrm{d}x}e^{ix} = \cos' x + i\sin' x = -\sin x + i\cos x = ie^{ix}$$
  - Hence, the *anti-derivative* $\int e^{ix}\,\mathrm{d}x$ of e^{ix} (the unique-up-to-a-constant function $f\colon \mathbb{R} \to \mathbb{C}$ whose derivative is e^{ix}) is $\frac{1}{i}e^{ix} + C = -ie^{ix} + C$, where $C \in \mathbb{C}$ is an arbitrary *complex* constant.
  - More generally, using that $e^{a+ib} = e^a \cdot e^{ib}$, we have that $\frac{\mathrm{d}}{\mathrm{d}x}e^{(a+ib)x} + (a+ib)e^{(a+ib)x}$ and $\int e^{(a+ib)x}\,\mathrm{d}x = \frac{1}{a+ib}e^{(a+ib)x} + C$.

2 Complex solutions to second-order equations

2.1 Example 11.2.1

- Now consider a general homogeneous second order linear equation $Ly = (D^2 + aD + b)y = 0$, with its characteristic equation $r^2 + ar + b = 0$.

  - We are perhaps still mainly interested in solutions $y\colon \mathbb{R} \to \mathbb{R}$, but now we can also try to find all solutions $y\colon \mathbb{R} \to \mathbb{C}$; and in this case, we can also consider equations with coefficients $a, b \in \mathbb{C}$.

- We now factor this polynomial as $(r - r_1)(r - r_2)$, with roots $-a \pm \frac{1}{2}\sqrt{a^2 - 4b}$ (where this square root is now possibly complex).

- We can thus factor the differential operator as $L = (D - r_1)(D - r_2)$.

- We will see that the above exponential multiplier method still works because $D(e^{rx}y) = e^{rx}(D+r)y$ as before, even for $r \in \mathbb{C}$.

- As a first example, consider $y'' + y = 0$, i.e., $(D^2+1)y = 0$, i.e., $(D-i)(D+i)y = 0$.
  - We would like to conclude from this that $(D+i)y = c_1 e^{ix}$ for some $c_1 \in \mathbb{C}$.
  - That is, we would like to say that $u = Ce^{ix}$ is the *general* solution $u\colon \mathbb{R} \to \mathbb{C}$ to the differential equation $u' = iu$.
  - Indeed, the usual method works: given any solution u, we have $\frac{\mathrm{d}}{\mathrm{d}x}(ue^{-ix}) = u'e^{-ix} - iue^{-ix} = 0$, hence $ue^{-ix} = C$ for some constant $C \in \mathbb{C}$, and hence $u = Ce^{ix}$.

- It now remains to solve $y' + iy = c_1 e^{ix}$, and for this, we again use exponential multipliers.
  - This equation is equivalent to $e^{ix}(y' + iy) = c_1 e^{2ix}$ (because we can multiply e^{-ix} to go back!), or in other words $\frac{\mathrm{d}}{\mathrm{d}x}(ye^{ix}) = c_1 e^{2ix}$.
    * Note that this use of the product rule is legitimate: it is just the ordinary product rule applied to each of the two components of the function $\mathbb{R} \to \mathbb{C}$ given by $x \mapsto y(x)e^{ix}$.
  - This is equivalent to $ye^{ix} = \frac{1}{2}c_1 e^{2ix} + c_2$ for some $c_2 \in \mathbb{C}$ and hence to

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

    for some $c_1, c_2 \in \mathbb{C}$; thus, this is the general solution.

- We can rewrite this as

$$\begin{aligned} y &= c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x) \\ &= (c_1 + c_2)\cos x + (c_1 - c_2) i \sin x \\ &= d_1 \cos x + d_2 \sin x, \end{aligned}$$

  where we have set $d_1 = c_1 + c_2$ and $d_2 = i(c_1 - c_2)$.
  - Since we can solve for c_1 and c_2 in terms of d_1 and d_2, this formula also expresses the general solution.
  - It is the general solution we might have expected for $y'' = -y$, except that now d_1 and d_2 may be *complex*.
  - The general *real* solution is obtained by restricting to the case $d_1, d_2 \in \mathbb{R}$.

2.2 Theorem 11.2.3

- The above example shows how the proof of Theorem 11.1.1 above can be adapted to the complex context, to yield the following statement:

- Given any $a, b \in \mathbb{C}$, the differential equation $y'' + ay' + by = 0$ has the general solution $y\colon \mathbb{R} \to \mathbb{C}$ given by

$$\begin{cases} y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, & r_1 \neq r_2 \\ y = c_1 x e^{r_1 x} + c_2 e^{r_2 x}, & r_1 = r_2, \end{cases}$$

  with $c_1, c_2 \in C$, where $r_1, r_2 \in \mathbb{C}$ are the roots of $r^2 + ar + v = 0$.

- In the case where $a, b \in \mathbb{R}$ and $a^2 - 4b < 0$, so that $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ for some $\alpha, \beta \in \mathbb{R}$, the general solution can also be written

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x,$$

  where $c_1, c_2 \in \mathbb{C}$. In this case the *real* solutions $y\colon \mathbb{R} \to \mathbb{R}$ are precisely those with $c_1, c_2 \in \mathbb{R}$.

- Moreover, the initial conditions $y(x_0) = y_0$ and $y'(x_0) = z_0$, for any $x_0 \in \mathbb{R}$ and $y_0, z_0 \in \mathbb{C}$ can always be satisfied by a unique choice of c_1 and c_2.

2.3 Example 11.2.2

- A generalization of the equation $y'' + y = 0$ from Example 11.2.1 is the equation $y'' + \omega^2 y = 0$, where $\omega \in \mathbb{R} - \{0\}$; this is called the *harmonic oscillator equation* with *angular frequency* ω.
- This has characteristic equation $r^2 + \omega^2 = 0$ with roots $r = \pm i\omega$, and hence general solution

$$y = c_1 e^{i\omega x} + c_2 e^{-i\omega x} = d_1 \cos \omega x + d_2 \sin \omega x.$$

- Such functions are called *harmonic oscillators*, and arise frequently in physics.

2.4 Example 11.2.3

- We summarize the form of the general solution to $y'' + ay' + by = 0$ with $a, b \in \mathbb{R}$.
- This has characteristic equation $r^2 + ar + b =$ with roots $r_1, r_2 = -a/2 \pm \sqrt{a^2 - 4b}/2$.
- We now consider three cases, based on the sign of the discriminant $a^2 - 4b$:
- If $a^2 - 4b > 0$, then r_1, r_2 are real and distinct, and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ with $c_1, c_2 \in \mathbb{R}$.
- If $a^2 - 4b < 0$, then r_1, r_2 are imaginary and distinct, and the general solution is $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$ with $c_1, c_2 \in \mathbb{R}$, where $\alpha = -a/2$, and $\beta = \sqrt{a^2 - 4b}/2$.
- If $a^2 - 4b = 0$ represents the dividing line between the above oscillatory and non-oscillatory behaviour; the general solution is $y = c_1 x e^{r_1 x} + c_2 e^{r_2 x}$.