Homework 3

Due: Wednesday, January 31

Math 435, Fall 2024

Problem 1. Spivak 3-14

Problems. **3-14.** Show that if $f,g\colon A \to \mathbf{R}$ are integrable, so is $f \cdot g$.

Hint: You may want to first prove it in the case $f, g \geq 0$. You can then use that $f = f_+ + f_- := \frac{1}{2}(f + |f|) + \frac{1}{2}(f - |f|)$ and $g = g_+ + g_- := \frac{1}{2}(g + |g|) + \frac{1}{2}(g - |g|)$ and apply the results you proved in the last homework.

Problem 2. Spivak 3-8

Problems. **3-8.** Prove that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ does not have content 0 if $a_i < b_i$ for each i.

Problem 3. Spivak 3-15

3-15. Show that if C has content 0, then $C \subset A$ for some closed rectangle A and C is Jordan-measurable and $\int_A \chi_C = 0$.

Problem 4. Spivak 3-26

3-26. Let $f\colon [a,b] \to \mathbf{R}$ be integrable and non-negative and let $A_f = \{(x,y)\colon a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$. Show that A_f is Jordan-measurable and has area $\int_a^b f$.

Problem 5. Spivak 3-32

3-32.* Let $f\colon [a,b] \times [c,d] \to \mathbf{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Prove *Leibnitz's rule:* $F'(y) = \int_a^b D_2 f(x,y)dx$. *Hint:* $F(y) = \int_a^b f(x,y)dx = \int_a^b \left(\int_c^y D_2 f(x,y)dy + f(x,c)\right)dx$. (The proof will show that continuity of $D_2 f$ may be replaced by considerably weaker hypotheses.)

Problem 6. Spivak 3-35

3-35.* (*a*) Let $g: \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation of one of the following types:

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = ae_j \end{cases}$$

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$$

$$\begin{cases} g(e_k) = e_k & k \neq i, j \\ g(e_i) = e_j \\ g(e_j) = e_i. \end{cases}$$

If U is a rectangle, show that the volume of $g(U)$ is $|\det g| \cdot v(U)$.

(b) Prove that $|\det g| \cdot v(U)$ is the volume of $g(U)$ for any linear transformation $g: \mathbf{R}^n \to \mathbf{R}^n$. *Hint:* If $\det g \neq 0$, then g is the composition of linear transformations of the type considered in (a).

Problem 7. Spivak 3-36

3-36. (Cavalieri's principle). Let A and B be Jordan-measurable subsets of $\mathbf{R}^3$. Let $A_c = \{(x,y): (x,y,c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Show that A and B have the same volume.