Homework 4

Due: Wednesday, February 7

Math 435, Fall 2024

Problem 1. Spivak 2-26 (the existence of bump functions)

2-26.* Let $$f(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1), \\ 0 & x \notin (-1,1). \end{cases}$$

(a) Show that $f\colon \mathbf{R} \to \mathbf{R}$ is a C^∞ function which is positive on $(-1,1)$ and 0 elsewhere.

(b) Show that there is a C^∞ function $g\colon \mathbf{R} \to [0,1]$ such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq \varepsilon$. *Hint:* If f is a C^∞ function which is positive on $(0,\varepsilon)$ and 0 elsewhere, let $g(x) = \int_0^x f / \int_0^\varepsilon f$.

(c) If $a \in \mathbf{R}^n$, define $g\colon \mathbf{R}^n \to \mathbf{R}$ by

$$g(x) = f([x^1 - a^1]/\varepsilon) \cdot \ldots \cdot f([x^n - a^n]/\varepsilon).$$

Show that g is a C^∞ function which is positive on

$$(a^1 - \varepsilon, a^1 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)$$

and zero elsewhere.

(d) If $A \subset \mathbf{R}^n$ is open and $C \subset A$ is compact, show that there is a non-negative C^∞ function $f\colon A \to \mathbf{R}$ such that $f(x) > 0$ for $x \in C$ and $f = 0$ outside of some closed set contained in A.

(e) Show that we can choose such an f so that $f\colon A \to [0,1]$ and $f(x) = 1$ for $x \in C$. *Hint:* If the function f of (d) satisfies $f(x) \geq \varepsilon$ for $x \in C$, consider $g \circ f$, where g is the function of (b).

Problem 2. Spivak 3-38 (an example showing why we need to demand convergence of $\sum_{\varphi\in\Phi} \int_A \varphi \cdot |f|$, and not just of $\sum_{\varphi\in\Phi} |\int_A \varphi \cdot f|$, in the definition (on Spivak p. 65) of $\int_A f := \sum_{\varphi\in\Phi} \int_A \varphi \cdot f$ for an open set A)

3-38. Let A_n be a closed set contained in $(n, n+1)$. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ for $x \notin$ any A_n. Find two partitions of unity Φ and Ψ such that $\sum_{\varphi \in \Phi} \int_{\mathbf{R}} \varphi \cdot f$ and $\sum_{\psi \in \Psi} \int_{\mathbf{R}} \psi \cdot f$ converge absolutely to different values.