Homework 6

Due: Wednesday, February 21

Math 435, Fall 2024

Problem 1. Let ω_1, ω_2 be (smooth) differential forms defined on $\mathbb{R}^n$, let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map. Prove that:

(i) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$ (assuming ω_1, ω_1 are both k-forms for some k)

(ii) $f^*(g \cdot \omega_1) = (g \circ f) \cdot f^*\omega_1$

(iii) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$

Problem 2. Spivak 4-14

4-14. Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function $c\colon [0,1] \to \mathbf{R}^n$. Define the **tangent vector** v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If $f\colon \mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.

Problem 3. Spivak 4-13 (b)

(b) If $f, g\colon \mathbf{R}^n \to \mathbf{R}$, show that $d(f \cdot g) = f \cdot dg + g \cdot df$.

Problem 4. Spivak 4-18

4-18. If $f\colon \mathbf{R}^n \to \mathbf{R}$, define a vector field grad f by

$$(\operatorname{grad} f)(p) = D_1 f(p) \cdot (e_1)_p + \cdots + D_n f(p) \cdot (e_n)_p.$$

For obvious reasons we also write grad $f = \nabla f$. If $\nabla f(p) = w_p$, prove that $D_v f(p) = \langle v, w \rangle$ and conclude that $\nabla f(p)$ is the direction in which f is changing fastest at p.

Problem 5. Spivak 4-19 parts (a) and (b).

4-19. If F is a vector field on $\mathbf{R}^3$, define the forms

$$\omega_F^1 = F^1\,dx + F^2\,dy + F^3\,dz,$$
$$\omega_F^2 = F^1\,dy \wedge dz + F^2\,dz \wedge dx + F^3\,dx \wedge dy.$$

(a) Prove that

$$df = \omega^1_{\operatorname{grad} f},$$
$$d(\omega_F^1) = \omega^2_{\operatorname{curl} F},$$
$$d(\omega_F^2) = (\operatorname{div} F)\,dx \wedge dy \wedge dz.$$

(b) Use (a) to prove that

$$\operatorname{curl} \operatorname{grad} f = 0,$$
$$\operatorname{div} \operatorname{curl} F = 0.$$