Homework 8

Due: Wednesday, March 20

Math 435, Fall 2024

Problem 1. Let $M \subset \mathbb{R}^n$ be a smooth k manifold given by $M = \{x \in \mathbb{R}^n \mid f(x) = y\}$, where $f \colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a smooth map for which y is a regular value.

Show that for any $x \in M$, the tangent space $\mathrm{T}_x M$ is the nullspace of the linear map $f_* \colon \mathbb{R}^n_x \to \mathbb{R}^k_y$:

$$\mathrm{T}_x M = \{v \in \mathbb{R}^n_x \mid f_*(v) = 0\}.$$

Problem 2. Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be manifolds, fix a point $y \in N$, and let $f \colon M \to N$ be the constant map $f(x) = y$.

Show that for any $x \in M$ the derivative $f_* \colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ of f at x is the zero map: $f_*(v) = 0$ for all $v \in \mathrm{T}_x M$.

Problem 3. Prove that the union $X = \{(x, y) \mid x = 0 \text{ or } y = 0\} \subset \mathbb{R}^2$ of axes in $\mathbb{R}^2$ is *not* a (smooth) 1-manifold.

Hint: If X were a 1-manifold, what would its tangent space $\mathrm{T}_0 X$ at the origin be?

Problem 4. Spivak 5-10

> **5-10. Suppose $\mathcal{C}$ is a collection of coordinate systems for M such that (1) For each $x \in M$ there is $f \in \mathcal{C}$ which is a coordinate system around x; (2) if $f, g \in \mathcal{C}$, then $\det (f^{-1} \circ g)' > 0$. Show that there is a unique orientation of M such that f is orientation-preserving if $f \in \mathcal{C}$.**

(Recall that what Spivak calls a "coordinate system" is what I have been calling a "parametrization".)

Problem 5. Let $\mathrm{S}^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and let $i \colon \mathrm{S}^2 \to \mathbb{R}^3$ be the inclusion map $i(x, y, z) = (x, y, z)$.

Let $\omega \in \Omega^2(\mathbb{R}^3)$ be the 2-form

$$\omega = \mathrm{d}x \wedge \mathrm{d}y$$

Show that the 2-form $i^*\omega \in \Omega^2(\mathrm{S}^2)$ on S^2 vanishes along the equator $E = \{(x, y, z) \in \mathrm{S}^2 \mid z = 0\}$, i.e., $(i^*\omega)(p) = 0$ for all $p \in E$.

Hint: Show that $(e_3)_p \in \mathrm{T}_p \mathrm{S}^2$ for all $p \in E$, where $e_3 = (0, 0, 1)$.