Homework 9
Due: Wednesday, March 27
Math 435, Fall 2024

Problem 1. If $M \subset \mathbb{R}^n$ is an oriented 1-manifold with volume form $\mathrm{d}V \subset \mathbb{R}^n$ and $\gamma\colon (a,b) \to M \cap U$ is an orientation-preserving diffeomorphism, show that

$$\mathrm{vol}(M \cap U) = \int_a^b |\gamma'(t)| \,\mathrm{d}t.$$

(Recall that, by definition, $\mathrm{vol}(M \cap U) = \int_{(a,b)} \gamma^*(\mathrm{d}V)$.)
Hint: We know $\gamma^* \mathrm{d}V$ must be of the form $f\,\mathrm{d}t$. What is f?

Problem 2. Spivak 5-31 parts (a) and (b).

5-31. Consider the 2-form ω defined on $\mathbf{R}^3 - 0$ by

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

(a) Show that ω is closed.
(b) Show that

$$\omega(p)(v_p, w_p) = \frac{\langle v \times w, p\rangle}{|p|^3}.$$

For $r > 0$ let $S^2(r) = \{x \in \mathbf{R}^3\colon |x| = r\}$. Show that ω restricted to the tangent space of $S^2(r)$ is $1/r^2$ times the volume element, and that $\int_{S^2(r)} \omega = 4\pi$. Conclude that ω is not exact.

Problem 3. Spivak 5-35. (Note that by "generalized divergence theorem", he just means the divergence theorem in $\mathbb{R}^n$ for general n, rather than just $n = 3$.)

5-35. Applying the generalized divergence theorem to the set $M = \{x \in \mathbf{R}^n\colon |x| \le a\}$ and $F(x) = x_x$, find the volume of $S^{n-1} = \{x \in \mathbf{R}^n\colon |x| = 1\}$ in terms of the n-dimensional volume of $B_n = \{x \in \mathbf{R}^n\colon |x| \le 1\}$. (This volume is $\pi^{n/2}/(n/2)!$ if n is even and $2^{(n+1)/2}\pi^{(n-1)/2}/1 \cdot 3 \cdot 5 \cdot \ldots \cdot n$ if n is odd.)

Problem 4. Spivak 5-36

5-36. Define F on $\mathbf{R}^3$ by $F(x) = (0,0,cx^3)_x$ and let M be a compact three-dimensional manifold-with-boundary with $M \subset \{x\colon x^3 \le 0\}$. The vector field F may be thought of as the downward pressure of a fluid of density c in $\{x\colon x^3 \le 0\}$. Since a fluid exerts equal pressures in all directions, we define the *buoyant force* on M, due to the fluid, as $-\int_{\partial M} \langle F, n\rangle\, dA$. Prove the following theorem.
Theorem (Archimedes). The buoyant force on M is equal to the weight of the fluid displaced by M.