Math 435 course outline

Spring 2024

Contents

Math 435: Lecture 1

January 8, 2024

Reference: Spivak, pp. 1-10

Topics:

- $\mathbb{R}^n$ (Spivak p. 1)
  - Vector space structure
  - Norm $\|x\|$ and inner product $\langle x, y\rangle$ and their basic properties (Spivak Theorems 1-1 and 1-2)
  - Linear maps $T\colon \mathbb{R}^n \to \mathbb{R}^m$, their representing matrices, composition.
- Subsets of Euclidean space
  - Closed and open rectangles in $\mathbb{R}^n$
  - Open and closed sets
  - Interior, exterior, and boundary points of a set
  - Open covers, compactness
  - Heine-Borel (compact if and only if closed and bounded – Spivak (1-3)-(1-7) and Problem 1-20)

Exercises:

- Spivak 1-7

1-7. A linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **norm preserving** if $|T(x)| = |x|$, and **inner product preserving** if $\langle Tx, Ty\rangle = \langle x, y\rangle$.

(a) Prove that T is norm preserving if and only if T is inner-product preserving.

(b) Prove that such a linear transformation T is 1-1 and T^{-1} is of the same sort.

Math 435: Lecture 2

January 10, 2024

Reference: Spivak, pp. 7-33

Topics:

- Heine-Borel (compact if and only if closed and bounded – Spivak (1-3)-(1-7) and Problem 1-20)
  - Spivak does the "$\Leftarrow$" direction in in three steps:
  - (i) A closed finite interval is compact
    (This part makes essential use of the *completeness* of $\mathbb{R}$ – i.e., the *least upper bound property.*)
  - (ii) The product of compact sets is compact; hence a closed rectangle is compact
    (This is proven using the so-called "tube lemma", Spivak 1-4)
  - (iii) A closed subset of a compact set is compact
    (This part is easy!)
  - The "$\Rightarrow$" direction is easy and is left as an exercise (Problem 1-20)
- A function $f\colon A \to \mathbb{R}^m$ (with $A \subset \mathbb{R}^n$) is continuous if and only if the preimage of any open set is open (Spivak Theorem 1-8)
  - Here, "continuous" means that $\lim_{x\to a} f(x) = f(a)$ for all $a \in A$, or equivalently: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \varepsilon$.
  - This is equivalent to each component of f being continuous (Spivak Problem 1-24).
- The continuous image of a compact set is compact (Spivak Theorem 1-9)
- Differentiation
  - Differentiability of functions $f\colon A \to \mathbb{R}^m$ in terms of linear approximation (Spivak Theorem 2-1 and the preceding discussion)
  - Derivatives in terms of component functions (Spivak Theorem 2-3 (3))
  - Jacobian matrix in terms of partial derivatives; continuously differentiable functions (Spivak Theorems 2-7 and 2-8)

Math 435: Lecture 3

January 12, 2024

Reference: Spivak, pp. 7-39

Topics:

- A little more on derivatives
  - Review of the three perspectives on the derivative: linear approximation, matrix of partial derivatives, directional derivative
  - $\mathcal{C}^\infty$ (smooth) functions and commutativity of partial derivatives (Spivak 2-5)
    Note: we will be mainly interested in smooth (hence continuously differentiable by Spivak Problem 2-1) functions.
  - Derivative at a maximum or minimum (Spivak 2-6)
  - Chain rule for derivatives (Spivak 2-2) and partial derivatives (Spivak 2-9)
- The all-important inverse function theorem (Spivak 2-11)
  - The easy direction:
    Exercise: suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and, for some $a \in \mathbb{R}^n$, there exists open subsets $V \subset \mathbb{R}^n$ containing $f(a)$ and $W \subset \mathbb{R}^n$ containing $f(a)$ such that $f\colon V \to W$ has a differentiable inverse $f^{-1}\colon W \to V$.
    Then $\det f'(a) \neq 0$, and in fact $(f^{-1})'\,(f(a)) = [f'(a)]^{-1}$.
  - Proof in the easy case $n = 1$.
  - Warning: inverse can still exist even if $\det f'(a) = 0$.
    Example: $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Math 435: Lecture 4

January 17, 2024

Reference: Spivak, pp. 40-43

Topics:

- Review of some properties of real numbers
  - Besides all of the usual properties of the real numbers (associativity, commutativity, distributivity, the properties of 1 and 0, and the basic properties of the ordering $a < b$), the real numbers have one more fundamental property:
  - The *least upper bound property*: any set of real numbers S which has an upper bound (i.e., a number M such that $x \leq M$ for all $x \in S$) has a *least upper bound* or *supremum* (i.e., an upper bound M such that $M \leq M'$ for any other upper bound M').
  - This implies the analogous "greatest lower bound" (or *infimum*) property. (**Exercise!**)
  - Some consequences:
  - (i) Any closed and bounded set of reals has a maximal element.
  - (ii) Any open interval of real numbers contains a rational number.
    (This uses three intermediate facts: (iii) the set of natural numbers is not bounded, (iv) for any positive real number, there is a smaller positive rational number, (v) if $b - a > 1$, then there is a rational number between a and b.)
  - The least upper bound property is also used in the proof other basic facts of calculus, such as the intermediate value theorem.
- Sample application of the inverse function theorem: the implicit function theorem (Spivak 2-12)
  - (The theorem is nice, but it is really the method of proof that is important, rather than theorem itself.)
  - The example of the circle
  - Proof in the case of functions $f\colon \mathbb{R}^2 \to \mathbb{R}$
  - Main idea: figure out how to turn f into a function $\mathbb{R}^2 \to \mathbb{R}^2$ with $\det f'(a, b) \neq 0$ so that you can apply the inverse function theorem.

Exercises:

(1) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and that the linear map $Df(a)$ is surjective for some $a \in \mathbb{R}^n$. Show that there is some open subset of $\mathbb{R}^m$ containing $f(a)$ which is in the image of f.

(2) Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and that the linear map $Df(a)$ is injective for some $a \in \mathbb{R}^n$. Show that there is some open subset of $\mathbb{R}^n$ containing a on which f is injective.

Math 435: Lecture 5

January 19, 2024

Reference: Spivak, pp. 34-49

Topics:

- A few more remarks on the inverse function theorem:
  - The theorem still holds if $\mathcal{C}^1$ is replaced by $\mathcal{C}^k$ for any k (including $k = \infty$): i.e., if the original function f is $\mathcal{C}^k$, then the inverse to f guaranteed by the theorem is also $\mathcal{C}^k$ (see Spivak, Addendum 1)c
  - Some useful terminology: a map $\mathbb{R}^m \supset U \xrightarrow{f} V \subset \mathbb{R}^n$ is a $\mathcal{C}^k$*-diffeomorphism* if it is $\mathcal{C}^k$ and has a $\mathcal{C}^k$-inverse. (Spivak p. 109)
    f is a *local* $\mathcal{C}^k$*-diffeomorphism at* $a \in U$ if there is some open set $a \in U' \subset U$ such that $f(U')$ is open and the restriction $f\colon U' \to f(U')$ is a $\mathcal{C}^k$-diffeomorphism.
  - Thus the inverse function theorem says: if $\mathrm{D}f(a)$ is invertible, then f is a local diffeomorphism at a.
  - Now an example: let $U = \mathbb{R} \times (0,1) \subset \mathbb{R}^2$ and $V = \{\vec{x} \in \mathbb{R}^2 \mid 0 < |\vec{x}| < 1\}$.
    Let $f\colon U \to V$ be defined by $f(x, y) = (y \cos x, y \sin x)$.
    We have

$$f'(x,y) = \begin{bmatrix} -y\sin x & \cos x \\ y\cos x & \sin x \end{bmatrix}$$

    and hence $\det f'(x, y) = -y(\sin^2 x + \cos^2 x) = -y \neq 0$.
    Hence by IFT, f is a local diffeomorphism at every point of U.
    However, f is clearly not a diffeomorphism, since it is not injective.
    Rather, $f\colon U \to f(U) = V - \{(y \cos t, y \sin y) \mid y \in (0, 1)\}$ is a diffeomorphism with $U = (t, t + 2\pi) \times (0, 1)$ for any $t \in \mathbb{R}$.
- Definition of the integral of a bounded function defined on a closed rectangle $A \subset \mathbb{R}^n$
  - Partitions of closed rectangles into subrectangles (Spivak p. 46)
  - Volumes of rectangles $v(R)$, and lower and upper sums $L(f, P) = \sum_S m_S(f) \cdot v(S)$ and $U(f, P) = \sum_S M_S(f) \cdot v(S)$ (p. 47)
  - If P' refines P, then $L(f, P) \leq L(F, P')$ and $U(F, P') \leq U(f, P)$. (Spivak 3-1)
  - If P and P' are any two partitions, then $L(f, P') \leq U(f, P)$ (Spivak 3-2)
  - $f\colon A \to \mathbb{R}$ is integrable if it is bounded and $\sup_P L(f, P) = \inf_P U(f, P)$.
    This number is called the *integral* of f over A, denoted $\int_A f$ or $\int_A f(x_1, \ldots, x^n)\, \mathrm{d}x^1 \cdots \mathrm{d}x^n$ (or $\int_a^b f$ in the case $n = 1$). (Spivak p. 48)
  - A bounded f is integrable if and only if for every $\varepsilon > 0$ there is a partition P with $U(f, P) - L(f, P)$ (Spivak 3-3)
  - Examples: (i) constant function and (ii) $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ defined by $f(x, y) = 0$ if $x \in \mathbb{Q}$ and $f(x, y) = 1$ else

Exercises:

(1) Spivak 3-1

Problems. **3-1.** Let $f\colon [0,1] \times [0,1] \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that f is integrable and $\int_{[0,1\times[0,1]} f = \frac{1}{2}$.

(2) Spivak 3-2

3-2. Let $f\colon A \to \mathbf{R}$ be integrable and let $g = f$ except at finitely many points. Show that g is integrable and $\int_A f = \int_A g$.

Math 435: Lecture 6

January 22, 2024

Reference: Spivak, pp. 50-51

Topics:

- Basic facts about integrals (Spivak, problems on p. 49 and Problem 3-14):
  - If f and g are integrable, then so is $f+g$, and $\int_A f+g = \int_A f + \int_A g$
  - If f is integrable, then so is $c \cdot f$, and $\int_A c \cdot f = c \cdot \int_A f$
  - If f and g are both integrable and $f \leq g$, then $\int_A f \leq \int_A g$
  - If f is integrable, then so is $f_+ = \max(0, f)$ and $f_- = \min(0, f)$
  - If f is integrable, then so is $|f|$, and $|\int_A f| = \int_A |f|$
  - If f and g are integrable, then so is $f \cdot g$.
  - The fundamental theorem of calculus: if f is continuous, then $F(x) = \int_a^x f$ is differentiable and $F' = f$.
- Measure zero
  - A set $A \subset \mathbb{R}^n$ has *measure 0* if for every $\varepsilon > 0$, there is a cover $\{U_1, U_2, \ldots,\}$ by closed (or equivalently, open) rectangles with $\sum_i v(U_i) < \varepsilon$.
  - A subset of a set with measure 0 has measure 0
  - Finite sets have measure 0.
  - Countable sets have measure 0 (take $v(U_i) < \varepsilon/2^i$)
  - $\mathbb{Q}$ has measure 0 (zigzag trick)
  - Theorem 3-4: A countable union of measure 0 sets has measure 0 (combine previous two tricks)

Exercise: Prove the above basic properties of integrals.

Math 435: Lecture 7

January 24, 2024

Reference: Spivak, pp. 51-56

Topics:

- *Content 0*: same thing as measure 0 but with finite covers.
  - Theorem 3-6: Compact and content 0 implies measure 0
    (Use open rectangles!)
  - Theorem 3-5: $[0, 1]$ does not have content 0 (hence does not have measure 0 since it is compact); in fact, any finite cover has $\sum_i U_i \geq b - a$
  - However, $[0, 1] \cap \mathbb{Q}$ does not have content 0 (since a finite union of closed intervals is closed, and $[0, 1] \cap \mathbb{Q}$ has closure $[0, 1]$).
  - The above basic facts about integrals
- Integrability and continuity
  - Theorem 3-8: A bounded function $f \colon A \to \mathbb{R}$ on a closed rectangle is integrable if and only if the set $B = \{x \mid f \text{ is not continuous at } x\}$ has measure zero.
  - Proof of the simple special case $B = \emptyset$ and under the assumption that f is *uniformly continuous*, meaning for all $\epsilon > 0$ there is a $\delta > 0$ such that $|\vec{x} - \vec{y}| < \delta(\varepsilon)$ implies $|f(\vec{x}) - f(\vec{y})| < \epsilon$ for any $x, y \in A$ (in fact, this is automatically true for any continuous function on a compact set):
    Fix $\epsilon > 0$. Then for any partition P such that each sub-rectangle has diameter $< \delta(\varepsilon/v(A))$, we have $U(f, P) - L(f, P) < (\varepsilon/v(A)) \cdot v(A) = \varepsilon$.
- Integration over more general bounded domains
  - The characteristic function $\chi_C(x)$ for $C \subset \mathbb{R}^n$ which is 1 for $x \in C$ and 0 else.
  - For $C \subset \mathbb{R}^n$ a bounded domain (so $C \subset A$ for some rectangle A) and a bounded function $f \colon C \to \mathbb{R}$, we define $\int_C f = \int_A f \cdot \chi_C$ (provided $f \cdot \chi_C$ is integrable – hence for example if both f and χ_C are integrable).

Exercises:

(1) Spivak 3-9

> **3-9.** (a) Show that an unbounded set cannot have content 0.
> (b) Give an example of a closed set of measure 0 which does not have content 0.

(2) Spivak 3-8

> ***Problems.*** **3-8.** Prove that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ does not have content 0 if $a_i < b_i$ for each i.

Math 435: Lecture 8

January 26, 2024

Reference: Spivak, pp. 56-62

Topics:

- Jordan-measurable sets
  - Theorem 3-9: the function χ_C is integrable if and only if the boundary of C has measure 0 (hence content 0, since a boundary is always closed, and C is bounded). (The reason is that the points of discontinuity of C are exactly the boundary points.)
  - Such a set C is called *Jordan-measurable* and $\int_C 1$ is its (n-dimensional) *content* or (n-dimensional) *volume* (or, in 1 and 2 dimensions, *length* or *area*).

  (Warning: not every open set is Jordan-measurable; later, we will introduce a generalized version of integration which is defined for bounded functions on any open set.)
- Fubini's theorem (3-10): if $f\colon A \times B \to \mathbb{R}$ is integrable, and $g_x\colon B \to \mathbb{R}$ defined by $g_x(y) = f(x, y)$ is integrable for all $x \in A$ and we define $G\colon A \to \mathbb{R}$ by $G(x) = \int_B g_x$, then we have $\int_{A\times B} f = \int_A G = \int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$.
  - Spivak proves a slightly stronger version of this theorem.
  - The hypothesis of the theorem holds whenever f is continuous.
  - Since integrability is not affected by changing the value of a function at finitely many points, the theorem still works if g_x is integrable for all but finitely many $x \in A$.
  - Of course, the same proof gives $\int_{A\times B} f = \int_B \left(\int_A f(x,y)\,\mathrm{d}x\right)\mathrm{d}y$.
  - Applying the theorem repeatedly, we have for $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f\colon A \to \mathbb{R}$ sufficiently nice (e.g., continuous) that $\int_A f = \int_{a_n}^{b_n} \left(\cdots \left(\int_{a_1}^{b_1} f(x^1, \ldots, x^n)\,\mathrm{d}x^1\right)\cdots\right)\mathrm{d}x^n$. Thus, in this case, multivariable integrals are reduced to single-variable ones.
  - The proof:
    Any partition P_A of A and P_B of B gives a partition $P_A \times P_B$ of $A \times B$ with sub-rectangles $S_A \times S_B$ where S_A and S_B are sub-rectangles of P_A and P_B.
    For any fixed S_A and $x \in S_A$, we have $m_{S_A\times S_B} \le m_{S_B}(g_x)$ and hence
    $\sum_{S_B} m_{S_A\times S_B}(f)v(S_B) \le \sum_{S_B} m_{S_B}(g_x)v(S_B) = L(g_x, P_B) \le \int_B g_x = G(x)$ and hence
    $\sum_{S_B} m_{S_A\times S_B}(f)v(s_B) \le m_{S_A}(G)$. Hence:
    $L(f, P_A \times P_B) = \sum_{S_A,S_B}(f)v(s_A)v(s_B) \le \sum_{S_A} m_{S_A}(G)v(S_A) = L(G, P_A) \le \int_A G$.
    The same argument gives $\int_A G \le U(f, P_A \times P_B)$, hence we have $L(f, P_A \times P_B) \le \int_A G \le U(f, P_A \times P_B)$ and it follows that $\int_{A\times B} f = \int_A G$.

Exercises:

(1) Spivak 3-15

> **3-15. Show that if C has content 0, then $C \subset A$ for some closed rectangle A and C is Jordan-measurable and $\int_A \chi_C = 0$.**

(2) Spivak 3-26

3-26. Let $f\colon [a,b] \to \mathbf{R}$ be integrable and non-negative and let $A_f = \{(x,y)\colon a \le x \le b \text{ and } 0 \le y \le f(x)\}$. Show that A_f is Jordan-measurable and has area $\int_a^b f$.

Math 435: Lecture 9

January 29, 2024

Reference: Spivak, pp. 63-64

Topics:

- Partitions of unity
  - For a set $A \subset \mathbb{R}^n$ and an open cover $\mathcal{O}$ of A, a $\mathcal{C}^\infty$ *partition of unity for* A is a collection Φ of smooth functions φ defined on an open neighbourhood of A such that
    (1) $0 \leq \varphi(x) \leq 1$ for $x \in A$
    (2) each $x \in A$ has an open neighbourhood V such that all but finitely φ are 0 on V
    (3) $\sum_{\varphi \in \Phi} \varphi(x) = 1$ for all $x \in A$.
    We say Φ is *subordinate* to $\mathcal{O}$ if moreover
    (4) for each $\varphi \in \Phi$ there is some $U \in \mathcal{O}$ such that $\varphi = 0$ outside some closed subset of U. (One says that φ is *supported* on U.)
  - Theorem 3-11: for any A and $\mathcal{O}$, there exists partition of unity for A subordinate to $\mathcal{O}$.
  - The main ingredient in the proof is the existence of *smooth bump functions*: for any open $U \subset \mathbb{R}^n$ and any compact $K \subset U$, there is a smooth function f supported in U with $f = 1$ on K.
    This is constructed in Spivak Problem 2-26.
    The starting point is $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.
    Then $g(x) = f(x)/(f(x) + f(1-x))$ is zero for $x \leq 0$ and 1 for $x \geq 1$.
  - The proof starts with a couple of reductions: first, any partition of unity for an open neighbourhood of A is also a partition of unity for A; hence, we may assume A is open. Next, we reduce to the case where A is compact, using that, for any open set U, there is a sequence of compact sets $K_1 \subset K_2 \subset \cdots$ with $\bigcup_i K_i = U$ and each K_i is contained in the interior of K_{i+1}.
  - Here is the proof when A is compact (using the existence of bump functions):
    For each $x \in A$, choose an open set $U_x \in \mathcal{O}$, and an open set V_x and compact set K_x with $x \in V_x \subset K_x \subset U_x$.
    Since A is compact, it is covered by finitely many of the V_x, say $V_{x_1}, \ldots, V_{x_N}$.
    Let ψ_i be a smooth bump function supported on U_{x_i} and equal to 1 on K_i.
    Then $\psi_1 + \cdots + \psi_N > 0$ on the open set $U = V_1 \cup \cdots \cup V_N$ containing A.
    Now define φ_i on U by letting $\varphi_i = \psi_i/(\psi_1 + \cdots \psi_N)$; note that φ_i may not be defined on all of U_i since the denominator can vanish somewhere on U_i.
    Hence, finally, let f be a bump function supported on U and equal to 1 on A. Then each $f\varphi_i$ is defined on all of U_i, and $\Phi = \{f \cdot \varphi_1, \ldots, f \cdot \varphi_N\}$ is the desired partition of unity.

Math 435: Lecture 10

January 31, 2024

Reference: Spivak, pp. 65-66

Topics:

- First application of partition of unity: integration on arbitrary open sets
  - Our current definition of $\int_A f$ for arbitrary A only works if A is bounded and bd$A$ has measure 0.
  - We would like a definition that works for arbitrary open sets A; but even a bounded open set may not be non-measure-0 boundary.
    (Example: Problem 3-11)
    We would also like to be able to integrate unbounded functions.
  - First, a reminder: a series $\sum_{i=1}^{\infty} a_i$ *converges absolutely* if $\sum_{i=1}^{\infty} |a_i|$ converges.
  - In this case, every reordering $\sum_{i=1}^{\infty} a_{k_i}$ of the series converges and has the same value.
  - (Moreover, the *Riemann series theorem* says that if $\sum_{i=1}^{\infty} a_i$ does *not* converge absolutely, then any $S \in \mathbb{R}$, there is a reordering a_{k_i} such that $\sum_{i=1}^{\infty} a_{k_i} = S$.)
  - Let A be open, let $\mathcal{O}$ be an open cover of A with $U \subset A$ for each $U \in \mathcal{O}$ and let Φ be a partition of unity subordinate to $\mathcal{O}$.
    Suppose $f : A \to \mathbb{R}$ is continuous outside of a set of measure 0, so that each $\int_A \varphi \cdot |f|$ exists.
  - We say that f is *integrable in the extended sense* if the series $\sum_{\varphi \in \Phi} \varphi \cdot \int |f|$ converges, and hence $\Sigma_{\varphi \in \Phi} \int_A \varphi \cdot f$ converges absolutely, and we define $\int_A f$ to be its sum.
  - (Though the convergence of $\sum_{\varphi \in \Phi} \varphi \cdot \int |f|$ is more than is needed to guarantee the absolute convergence of $\Sigma_{\varphi \in \Phi} \int_A \varphi \cdot f$, it *is* needed to ensure that this value is independent of the chosen partition of unity Φ, as is shown in Problem 3-38.)
  - Theorem 3-12:
    (1) $\int_A f$ is independent of the cover $\mathcal{O}$ and the partition of unity Φ.
    (2) If A and f are bounded (and A is open and f is continuous outside a set of measure 0), then $\int_A f$ exists.
    (3) This agrees with the old definition of $\int_A f$ when they are both defined.

Exercises:

(1) Spivak 3-36

3-36. (Cavalieri's principle). Let A and B be Jordan-measurable subsets of $\mathbf{R}^3$. Let $A_c = \{(x,y) : (x,y,c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Show that A and B have the same volume.

(2) Spivak 3-38

3-38. Let A_n be a closed set contained in $(n, n+1)$. Suppose that $f: \mathbf{R} \to \mathbf{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f = 0$ for $x \notin$ any A_n. Find two partitions of unity Φ and Ψ such that $\Sigma_{\varphi \in \Phi} \int_{\mathbf{R}} \varphi \cdot f$ and $\Sigma_{\psi \in \Psi} \int_{\mathbf{R}} \psi \cdot f$ converge absolutely to different values.

Math 435: Lecture 11

February 2, 2024

Reference: Spivak, pp. 66-76

Topics:

- Change of variables
  - Change of variables in one dimension: $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$ ("set $u = g(x)$, then $\mathrm{d}u = g'(x)\,\mathrm{d}x$").
    (Suppose f is continuous and g is continuously differentiable.)
  - More concisely: $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$.
  - Proof: if $F' = f$, then $(F \circ g)' = (f \circ g) \cdot g'$, so the left side $F(g(b)) - F(g(a))$ and the right side is $(F \circ g)(b) - (F \circ g)(a)$.
  - Assuming $g\colon (a,b) \to (g(a), g(b))$ is a diffeomorphism, this can be written as $\int_{g((a,b))} f = \int_{(a,b)} (f \circ g) \cdot |g'|$.
  - In higher dimensions, we have:
    Theorem 3-13: if $A \subset \mathbb{R}^n$ is open and $g\colon A \to g(A) \subset \mathbb{R}^n$ is a diffeomorphism, and $f\colon g(A) \to \mathbb{R}$ is integrable, then $\int_{g(A)} f = \int_A (f \circ g)|\det g'|$.
  - Example: integrating in polar coordinates
  - The proof has several steps.
  - One first uses partitions of unity to reduce to the simple case is in which (i) A is a rectangle, and (ii) f is the constant function 1.
  - This case is proven by induction on n, the base case $n = 1$ being already known.
    For the induction step when $n > 1$, there is a further reduction using the inverse function theorem to the case in which (iii) $g^n(x) = x^n$, hence $g(x) = (g^1(x), \dots, g^{n-1}(x), x^n)$.
    Now the claim follows from Fubini's theorem. Write $A = B \times [a_n, b_n]$ and define $g_{x^n}(x^1, \dots, x^{n-1}) = (g^1(x^1, \dots, x^n), \dots, g^{n-1}(x^1, \dots, x^n))$ for each $x^n \in [a,b]$, we have by induction that $\int_{g(B)} 1 = \int_B |\det g'_{x^n}|$.
    But we also have $\det g'_{x^n}(x^1, \dots, x^{n-1}) = \det g'(x^1, \dots, x^n)$ and hence, by Fubini:
$$\begin{aligned}
\int_{g(A)} 1 &= \int_{[a_n,b_n]} \int_{g(B \times x^n)} 1 \\
&= \int_{[a_n,b_n]} \Big(\int_{g_{x^n}(B)} 1 \Big)\,\mathrm{d}x^n \\
&= \int_{[a_n,b_n]} \Big(\int_B |\det g'_{x^n}| \Big)\,\mathrm{d}x^n \\
&= \int_{[a_n,b_n]} \Big(\int_B |\det g'(x^1, \dots, x^n)|\,\mathrm{d}x^1 \dots \mathrm{d}x^n \Big)\,\mathrm{d}x^n \\
&= \int_A |\det g'|.
\end{aligned}$$
  - It remains to explain how the reductions (i), (ii), (iii) are carried out.

- The reduction (i) involves a computation which looks like

$$\int_{g(A)} f = \sum_{\varphi\in\Phi} \int_{g(A)} \varphi \cdot f = \sum_{\varphi\in\Phi} \int_A (\varphi\circ g)\cdot(f\circ g)\cdot|\det g'| = \int_A (f\circ g)\cdot|\det g'|$$

  where Φ is a partition of unity (which we can take to be subordinate to a cover by rectangles).
  In fact, this shows that it even suffices to prove the theorem in an *arbitrarily small* rectangle around each point in A.
- The reduction (ii) is a direct computation from the definition of the integral using, roughly speaking, that that $\sum_S m_S(f)\chi_S \le f \le \sum_S M_S(f)\chi(S)$ for any partition P.
- For the reduction (iii), an easy computation shows that, if the theorem holds for a given diffeomorphism $g_1\colon A\to g_1(A)$ and for a second diffeomorphism $g_2\colon g_1(A)\to g_2(g_1(A))$, then it is also true for $g_2\circ g_1\colon A\to g_2(g_1(A))$.
  Now fix $g\colon A\to g(A)$, and $a\in A$.
  If T is the linear transformation $Dg(a)$, then $(T^{-1}\circ g)'(a) = I$ by the chain rule, and it suffices to prove the claim for $T^{-1}\circ g$ since $g = T\circ(T^{-1}\circ g)$, and since we already proved the theorem for linear transformations.
  In other words, we may assume $g'(a) = I$.
  Now define $h\colon A\to\mathbb{R}^n$ by $h(x) = (g^1(x),\dots,g^{n-1}(x),x^n)$. Then $\det h'(a) = \det I \neq 0$, hence $h\colon U\to h(U)$ is a diffeomorphism on some open neighbourhood U around a. Defining $k\colon h(U)\to\mathbb{R}^n$ by $k(y) = (y^1,\dots,y^{n-1},g^n(h^{-1}(y)))$, we then have $g = k\circ h$, and hence it suffices to prove the claim for h and k, which are both of the desired form.

- Multilinear maps/tensors
  - A multilinear map on a vector space V (over $\mathbb{R}$) is a function $T\colon V^k\to\mathbb{R}$ which is linear in the i-th input whenever all the other inputs are held constant (for any $i = 1,\dots,k$). In other words, given any $v_j\in V$ for all $j\neq i$, the function $V\to\mathbb{R}$ given by $v_i\mapsto T(v_1,\dots,v_n)$ is linear.
  - Spivak also calls a multilinear map $V^k\to\mathbb{R}$ a *k-tensor*. (Note: it would be more usual to call it a *$(k,0)$-tensor*, but this abbreviation is reasonable since we will not be dealing with (k,l)-tensors for any $l>0$.)
  - $\mathcal{T}^k(V)$ is the set of all tensors over V; it is a vector space under pointwise addition and scalar multiplication.
  - There is an operation $\otimes\colon \mathcal{T}^k(V)\times\mathcal{T}^l(V)\to\mathcal{T}^{k+l}(V)$ defined by $(S\otimes T)(v_1,\dots,v_{k+l}) = S(v_1,\dots,v_k)\cdot T(v_{k+1},\dots,v_{k+l})$.
  - **Exercise:** The operation $\otimes$ is bilinear and associative.
  - By associativity, we may as usual omit parentheses and write $S_1\otimes\cdots\otimes S_m$.

Math 435: Lecture 12

February 5, 2024

Reference: Spivak, pp. 76-77

Topics:

- Some example of vector spaces
  - Main example: linear subspaces of $\mathbb{R}^n$
  - The set of all $\mathcal{C}^k$ functions on a set $U \subset \mathbb{R}^n$ (this is infinite-dimensional!)
  - The set of all polynomials in n variables (also infinite-dimensional)
  - The set of all degree d polynomials in n variables (finite-dimensional)
  - Example from last time: the set $\mathcal{T}^k V$ of all k-tensors on V, i.e., multilinear maps $V^k \to \mathbb{R}$
- The axioms for a vector space:
  (i) associativity and commutativity of addition
  (ii) existence of the zero vector
  (iii) existence of additive inverses
  (iv) associativity of scalar multiplication
  (v) distributivity of scalar multiplication
  (vi) $1 \cdot v = v$
- From last time: **Exercise:** The operation $\otimes$ is bilinear and associative.
- By associativity, we may as usual omit parentheses and write $S_1 \otimes \cdots \otimes S_r$.
- The dual space and tensors
  - Note: $\mathcal{T}^1(V)$ is just the dual space V^*.
  - Recollection of dual bases: if V is a (finite-dimensional) vector space over $\mathbb{R}$, then for any basis $v_1, \ldots, v_n$ of V, there is a unique basis $\varphi_1, \ldots, \varphi_n$ of V^* with $\varphi_i(v_j) = \delta_{ij}$ called the *dual basis*.
    (And conversely, any basis of V^* is dual to a unique basis of V.)
    (δ_{ij} is the *Kronecker delta symbol*, which is defined to be 1 if $i = j$ and 0 if $i \neq j$.)
  - Theorem 4-1: if $v_1, \ldots, v_n$ is a basis for V with dual basis $\varphi_1, \ldots, \varphi_n$ of V^*, then the set of all k-fold tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ (with $i_1, \ldots, i_k \in \{1, \ldots, n\}$) is a basis for $\mathcal{T}^k(V)$, which therefore has dimension n^k.
  - The proof:
    First observation: $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, \ldots, v_{j_k}) = \delta_{i_1,j_1} \cdots \delta_{i_k,j_k}$.
    Second observation: $T = \sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1}, \ldots, v_{i_k})\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$
    (Applying both sides to an arbitrary input $(w_1, \ldots, w_k)$ and expressing each w_i as $w_i = \sum_{j=1}^{n} a_{i,j} v_j$, we get $\sum_{j_1,\ldots,j_k=1}^{n} a_{1,j_1}, \ldots, a_{k,j_k} T(v_{j_1}, \ldots, v_{j_k})$).
    Third observation: if $\sum_{i_1,\ldots,i_k=1}^{n} a_{i_1,\ldots,i_k}\, \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0$, then by applying this to $v_{i_1}, \ldots v_{i_k}$, we obtain $a_{i_1,\ldots,i_k} = 0$.
  - Given a linear map $f\colon V \to W$, there is an induced map $f^*\colon \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ given by $f^*T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$.

Exercises:

- Check that $\mathcal{T}^k(V)$ satisfies the axioms of a vector space.
- The operation $\otimes$ is bilinear and associative.
- $f^*(S \otimes T) = f^*S \otimes f^*T$

Math 435: Lecture 13

February 7, 2024

Reference: Spivak, pp. 77-79

Topics:

- Inner products are examples of 2-tensors
  - An inner product on a vector space V is a symmetric, positive-definite bilinear map $T \in \mathcal{T}^2(V)$.
  - Any (finite-dimensional) inner product space has an orthonormal basis (by the Gram-Schmidt process), and is thus isomorphic to $\mathbb{R}^n$ with the standard inner product.
- Alternating k-tensors
  - A k-tensor ω is *alternating* if switching two of the arguments in $\omega(v_1, \ldots, v_k)$ changes the sign of the result.
  - Example: $\det \in \mathcal{T}^n(\mathbb{R}^n)$; it is the *unique* alternating n-tensor with $\det(e_1, \ldots, e_n) = 1$.
  - We write $\Lambda^k(V) \subset \mathcal{T}^k(V)$ for the subspace consisting of alternating k-tensors.
  - Recall the sign $\operatorname{sgn}\sigma$ of a permutation $\sigma \in S_k$ of $\{1, \ldots, k\}$, which is $(-1)^N$ where N is the number of pairs $1 \leq i < j \leq k$ with $\sigma(j) < \sigma(i)$ (or equivalently, the number of transpositions performed to produce σ).
  - For $T \in \mathcal{T}^k(V)$, we define $\operatorname{Alt}(T)$ by $\operatorname{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn}\sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$. Here, S_{k+l} is the set of permutations of $\{1, \ldots, k+l\}$, i.e., bijections $\{1, \ldots, k+l\} \to \{1, \ldots, k+l\}$.
  - (Compare: $\det A = \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn}\sigma \cdot A_{1,\sigma(1)}, \ldots, A_{k,\sigma(k)}$.)
  - Theorem 4-3: (1) $\operatorname{Alt}(T) \in \Lambda^k(V)$, (2) if $\omega \in \Lambda^k(V)$, then $\operatorname{Alt}(\omega) = \omega$, (3) $\operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$.
- The wedge product
  - $\wedge\colon \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is defined by $\omega \wedge \eta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta)$.
  - The strange coefficient makes various formulas work out nicer later on.
    For now, note that for $\omega, \eta \in \Lambda^1(V) = \mathcal{T}^1(V) = V^*$, we have $(\omega \wedge \eta)(u, v) = \omega(u)\eta(v) - \eta(u)\omega(v)$.

Math 435: Lecture 14

February 9, 2024

Reference: Spivak, pp. 79-84

Topics:

- The wedge product
  - $\wedge\colon \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is defined by $\omega \wedge \eta = \frac{k!l!}{(k+l)!}\mathrm{Alt}(\omega \otimes \eta)$.
  - The strange coefficient makes various formulas work out nicer later on.
    For now, note that for $\omega, \eta \in \Lambda^1(V) = \mathcal{T}^1(V) = V^*$, we have $(\omega\wedge\eta)(u,v) = \omega(u)\eta(v) - \eta(u)\omega(v)$.
  - **Exercise:** Some basic facts about the wedge product:
    (i) $\wedge$ is bilinear
    (ii) $\wedge$ is "graded-commutative": $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$
    (iii) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$
  - $\wedge$ is also associative; in fact, we have (Theorem 4-4): $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$
  - The proof is more difficult then the above facts; it requires two lemmas:
  - Lemma 1: If $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^l(V)$ and $\mathrm{Alt}(S) = 0$, then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$.
  - Lemma 2: $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$.
  - The theorem follows from Lemma 2 which follows from Lemma 1
  - Proof of Lemma 1 when $k = 2$:
    Suppose $S \in \mathcal{T}^2(V)$ and $0 = \mathrm{Alt}(S)(v,w) = \frac{1}{2}(S(v,w) - S(w,v))$ for all v, w.
    We have $\mathrm{Alt}(S \otimes T)(v_1, \ldots, v_{l+2}) = \sum_\sigma \operatorname{sgn}\sigma \cdot S(v_{\sigma 1}, v_{\sigma 2})T(v_{\sigma 3}, \ldots, v_{\sigma(l+2)})$.
    But now for every σ there is an accompanying permutation $\sigma' = \sigma(12)$ with the opposite sign; since $S(v,w) = S(w,v)$, the sum breaks up into two parts which are the same but with opposite sign.
  - Again, by associativity, we may as usual omit parentheses and write $\omega_1 \wedge \cdots \wedge \omega_r$.
- The dimension of $\Lambda^k(V)$.
  - Theorem 4-5: if $v_1, \ldots, v_n$ is a basis for V with dual basis $\varphi_1, \ldots, \varphi_n$ of V^*, then the set of all k-fold wedge products $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$ is a basis for $\Lambda^k(V)$, which therefore has dimension $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
  - Proof: given $\omega \in \Lambda^k(V) \subset \mathcal{T}^k(V)$, write $\omega = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$.
    Thus $\omega = \mathrm{Alt}(\omega) = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k}\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$.
    But now $\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$ is some multiple of $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$, which is zero if any two i_j are equal, and otherwise is ± 1 times $\varphi_{i'_1} \wedge \cdots \wedge \varphi_{i'_k}$ where $i'_1 < \ldots < i'_k$.
    Thus our putative basis is a spanning set.
    The proof of independence is as above, using that $\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}(v_{j_1}, \ldots v_{j_k}) = \delta_{i_1 j_1} \cdots \varphi_{i_k j_k}$ if $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$:
    If $0 = \omega = \sum_{i_1 < \ldots < i_k} a_{i_1,\ldots,i_k}\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ then $0 = \omega(v_{i_1}, \ldots, v_{i_k}) = a_{i_1,\ldots,i_k}$.
- Alternating n-tensors on an n-dimensional space.
  - If $\dim V = n$, then $\dim \Lambda^n(V) = 1$, so all n-tensors are multiple of any non-zero one.
  - Hence any $\omega \in \Lambda^n(\mathbb{R}^n)$ is a multiple of det.

- Theorem 4-6: If $v_1, \dots, v_n$ is a basis for V and $\omega \in \Lambda^n(V)$, then for any n vectors $w_i = \sum_{i=1}^n a_{ij} v_j$, we have $\omega(w_1, \dots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \dots, v_n)$.
- For the proof, define $\eta \in \mathcal{T}^n(\mathbb{R}^n)$ by $\eta((a_{11}, \dots, a_{1n}), \dots, (a_{n1}, \dots, a_{nn}) = \omega(\sum_j a_{1j} v_j, \dots, \sum_j a_{nj} v_j)$. Then $\eta \in \Lambda^n(\mathbb{R}^n)$, so $\eta = \lambda \cdot \det$ for some $\lambda \in \mathbb{R}$, and $\lambda = \eta(e_1, \dots, e_n) = \omega(v_1, \dots, v_n)$.
- (Alternatively, since V is isomorphic to $\mathbb{R}^n$, it suffices to prove it in the latter case, in which case we know $\omega = c \cdot \det$ for some $c \in \mathbb{R}$, and in this case the claim follows right away if we assume the multiplicativity of the determinant: $\det(A \cdot B) = \det(A) \cdot \det(B)$.)

Exercises:

- The above basic facts about the wedge product
- Spivak 4-1

Problems. **4-1.*** Let $e_1, \dots, e_n$ be the usual basis of $\mathbf{R}^n$ and let $\varphi_1, \dots, \varphi_n$ be the dual basis.

(a) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \dots, e_{i_k}) = 1$. What would the right side be if the factor $(k + l)!/k!l!$ did not appear in the definition of $\wedge$?

(b) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \dots, v_k)$ is the determinant of the $k \times k$ minor of $\begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k \end{pmatrix}$ obtained by selecting columns $i_1, \dots, i_k$.

- More generally, show that for any $\varphi_1, \dots, \varphi_k \in V^*$ and any $v_1, \dots, v_k \in V$, we have $\omega_1 \wedge \dots \wedge \omega_k(v_1, \dots, v_k) = \det\left([\varphi_i(v_j)]_{i,j=1}^k\right)$.

Math 435: Lecture 15

February 12, 2024

Reference: Spivak, pp. 86-89

Topics:

- Orientations
  - By the theorem, any non-zero $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n)$ for any basis $v_1, \ldots, v_n$ (since by definition we must have $\omega(w_1, \ldots, w_n)$ for *some* vectors $w_1, \ldots, w_n$).
  - Thus, the set of bases of V is split into two subsets: those with $\omega(v_1, \ldots, v_n) > 0$ and those with $\omega(v_1, \ldots, v_n) < 0$.
  - Given two bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ with $w_i \sum_j a_{ij} v_j$, they will be in the same subset if and only if $\det(a_{ij}) > 0$.
  - This condition is independent of ω and always separates the bases of V into two subsets; each of these subsets is called an *orientation* of V.
  - For a basis $v_1, \ldots, v_n$, we write $[v_1, \ldots, v_n]$ for the orientation to which it belongs, and the other orientation is denoted $-[v_1, \ldots, v_n]$.
  - On $\mathbb{R}^n$, we define the *usual orientation* (or *standard orientation*) to be $[e_1, \ldots, e_n]$.
- Volume element
  - The characterization of $\det \in \Lambda^n(\mathbb{R}^n)$ by the property $\det(e_1, \ldots, e_n) = 1$ is not available in a general vector space V since there is no "standard basis".
  - But now suppose V has an inner product T, and consider orthonormal bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$.
  - If $w_i = \sum_j a_{ij} v_j$, then $A = (a_{ij})$ is an orthogonal matrix: $\delta_{ij} = T(w_i, w_j) = \sum_k a_{ik} a_{jk}$, or in other words $A^\top = A^{-1}$; hence $\det A = \pm 1$.
  - Thus, if $\omega(v_1, \ldots, v_n) = \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$.
  - If we moreover have an orientation μ, then there is a unique ω with $\omega(v_1, \ldots, v_n) = 1$ for an orthonormal oriented basis $[v_1, \ldots, v_n] = \mu$.
  - This is called the *volume element* of V determined by T and μ.
  - In $\mathbb{R}^n$, det is the volume element determined by the standard inner product and orientation.
  - The name comes from the fact that $\det(v_1, \ldots, v_n)$ is the volume of the parallelepiped spanned by $v_1, \ldots, v_n$.
- Tangent vectors
  - For fixed $p \in \mathbb{R}^n$, the set of all pairs (p, v) with $v \in \mathbb{R}^n$ is denoted $\mathbb{R}^n_p$ and is called the *tangent space to* $\mathbb{R}^n$ *at* p.
  - Of course, $\mathbb{R}^n_p$ is in bijection with $\mathbb{R}^n$ itself, and therefore is a vector space (and has a standard basis, standard inner product, standard orientation, etc.).
  - We write v_p for (p, v).
  - The *endpoint* of v_p is the point $p + v$.
- Vector fields
  - A *vector field* on $\mathbb{R}^n$ is a function F on $\mathbb{R}^n$ such that $F(p) \in \mathbb{R}^n_p$ for each p.
  - The *component functions* $F^1, \ldots, F^n \colon \mathbb{R}^n \to \mathbb{R}$ of F are given by $F(p) = F^1(p) \cdot (e_1)_p + \cdots + F^n(p) \cdot (e_n)_p$.
  - We say that F is a $\mathcal{C}^k$ vector field if each F^i is a $\mathcal{C}^k$ function.

  - Given vector fields F, G and a smooth function f, we can define vector fields $F+G$ and $f \cdot F$, and a function $\langle F, G\rangle$ pointwise: $(F+G)(p) = F(p)+G(p)$ and so on.
- Divergence and curl
  - The *divergence* $\mathrm{div}F$ of a vector field is $\sum_{i=1}^n D_i F^i$.
  - If we consider the "vector of differential operators" $\nabla = \sum_{i=1}^n D_i \cdot e_i = (D_1, \dots, D_n)$, we can write $\mathrm{div}F = \langle \nabla, F\rangle$.
  - Similarly, in $\mathbb{R}^3$, we can write $(\nabla \times F)$; this is called the *curl* of F.
  - The names "divergence" and "curl" comes from physics you may have seen; we will discuss it later.
- Differential forms
  - A *differential form of degree* k (or just k*-form*) on $\mathbb{R}^n$ is a function ω with $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ for each $p \in \mathbb{R}^n$.
  - If $\varphi_1(p), \dots, \varphi_n(p)$ is dual basis to $(e_1)_p, \dots, (e_n)_p$, we can write
$$\omega(p) = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\dots,i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)]$$
  - We say that ω is a $\mathcal{C}^l$ k-form if each $\omega_{i_1,\dots,i_k}$ is $\mathcal{C}^l$. As usual, we only really care about the case $l = \infty$.
  - We can define the sum $\omega+\eta$, multiple $f \cdot \omega$, and $\omega \wedge \eta$ of forms pointwise: $(\omega+\eta)(p) = \omega(p) + \eta(p)$ and so on.
  - We also consider a function as a 0-form and write $f \wedge \omega$ for $f \cdot \omega$.

Exercises:

- Prove (2)-(4) of Theorem 4-8
- Spivak 4-14

4-14. **Let c be a differentiable curve in $\mathbf{R}^n$, that is, a differentiable function $c\colon [0,1] \to \mathbf{R}^n$. Define the tangent vector v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \dots, (c^n)'(t))_{c(t)}$. If $f\colon \mathbf{R}^n \to \mathbf{R}^m$, show that the tangent vector to $f \circ c$ at t is $f_*(v)$.**

Math 435: Lecture 16

February 14, 2024

Reference: Spivak, pp. 89-90

Topics:

- Differential of a function
  - If $f\colon \mathbb{R}^n \to \mathbb{R}$, we have $\mathrm{D}f(p) \in \Lambda^1(\mathbb{R}^n)$; using the correspondence $\mathbb{R}^n_p \cong \mathbb{R}^n$, we thus obtain a 1-form $\mathrm{d}f$:
  $$\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v).$$
  - We write $x^i\colon \mathbb{R}^n \to \mathbb{R}$ for the function returning the i-th coordinate of a point.
  - Warning: there is potential for confusion: sometimes, we write $(x^1, \ldots, x^n)$ for the coordinates of a *given* point, so each x^i is a number. Now, we are writing x^i for the coordinate *function*, so $x^i(a^1, \ldots, a^n) = a^i$.
  - We now consider the 1-form $\mathrm{d}x^i$. We have
  $$\mathrm{d}x^i(p)(v_p) = \mathrm{D}x^i(p)(v) = v^i.$$
  - Hence $\mathrm{d}x^1(p), \ldots, \mathrm{d}x^n(p)$ is the dual basis to $(e_1)_p, \ldots, (e_n)_p$, and we can write any k-form ω as
  $$\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1,\ldots,i_k}(p)\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$
  - Theorem 4-7: if $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable, then
  $$\mathrm{d}f = \frac{\partial f}{\partial x^1} \cdot \mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^n} \cdot \mathrm{d}x^n.$$
  The proof: $\mathrm{d}f(p)(v_p) = \mathrm{D}f(p)(v) = \sum_{i=1}^n v^i \cdot \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^n \mathrm{d}x^i(p)(v_p) \cdot \frac{\partial f}{\partial x^i}(p)$.
- Pullbacks of differential forms
  - Given a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, we have $\mathrm{D}f(p)\colon \mathbb{R}^n \to \mathbb{R}^m$; thus identifying $\mathbb{R}^n, \mathbb{R}^m$ with $\mathbb{R}^n_p, \mathbb{R}^m_p$, we obtain a linear map $f_*\colon \mathbb{R}^n_p \to \mathbb{R}^m_p$:
  $$f_*(v_p) = \big(\mathrm{D}f(p)(v)\big)_{f(p)}$$
  (In an exercise, you will prove an important alternative characterization of f_*: each $v \in \mathbb{R}^n_p$ is the tangent vector $\gamma'(0)$ to a curve γ through p; and $f_* v \in \mathbb{R}^m_{f(p)}$ is just the tangent vector $(f \circ \gamma)'(p)$ of $f \circ \gamma$ through $f(p)$.)
  - We define the *pullback* of a k-form ω on $\mathbb{R}^m$ to be k-form $f^*\omega$ on $\mathbb{R}^n$ given by $(f^*\omega)(v_1, \ldots, v_k) = \omega\big(f(p)\big)(f_* v_1, \ldots, f_* v_k)$.
  - Theorem 4-8:
  (1) $f^*\,\mathrm{d}x^i = \mathrm{d}f^i$ (and more generally: $f^*\,\mathrm{d}g = \mathrm{d}(g \circ f)$)
  (2) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
  (3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$
  (4) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$
  - Part (1) follows from the chain rule
  - **Exercise:** Prove (2)-(4)
  - Example: suppose we have open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$, a differential form α defined on V, and a smooth function $f\colon U \to V$.

– It is helpful to use different notation for the coordinates on U and V; let us write $x, y \colon U \to \mathbb{R}$ for the coordinate functions on U and $X, Y, Z \colon V \to \mathbb{R}$ for the coordinates on V.

  Now suppose

$$\alpha = P(X,Y,Z)\,\mathrm{d}X \wedge \mathrm{d}Y + Q(X,Y,Z)\,\mathrm{d}Y \wedge \mathrm{d}Z$$

  Then

$$\begin{aligned} f^*\alpha &= (P \circ f)[f^*(\mathrm{d}X^1) \wedge f^*(\mathrm{d}X^2)] + (Q \circ f)[f^*(\mathrm{d}X^2) \wedge f^*(\mathrm{d}X^3)] \\ &= (P \circ f)[\mathrm{d}f^1 \wedge \mathrm{d}f^2] + (Q \circ f)[\mathrm{d}f^2 \wedge \mathrm{d}f^3] \end{aligned} \tag{1}$$

– What this means in practice is that we "express X, Y, Z in terms of x, y and substitute".

  Example: suppose

$$\begin{aligned} f(x,y) &= (x \sin y, xy, ye^x) \\ P(X,Y,Z) &= X + Y \\ Q(X,Y,Z) &= X + Z \end{aligned}$$

  so that

$$\alpha = (X+Y)\,\mathrm{d}X \wedge \mathrm{d}Y + (X+Z)\,\mathrm{d}Y \wedge \mathrm{d}Z.$$

  Now we perform the substitution $X = f^1(x,y) = x \sin y$, $Y = f^2(x,y) = xy$, $Z = f^3(x,y) = ye^x$, and obtain:

$$f^*\alpha = (x \sin y + xy)\,\mathrm{d}(x \sin y) \wedge \mathrm{d}(xy) + (x \sin y + ye^x)\,\mathrm{d}(xy) \wedge \mathrm{d}(ye^x).$$

  Notice how this corresponds precisely to the expression (1).

  To finish working this out, we should compute each of the differentials. For example:

$$\begin{aligned} \mathrm{d}(x \sin y) \wedge \mathrm{d}(xy) &= (\sin y\,\mathrm{d}x + x \cos y\,\mathrm{d}y) \wedge (x\,\mathrm{d}y + y\,\mathrm{d}x) \\ &= x \sin y\,\mathrm{d}x \wedge \mathrm{d}y + xy \cos y\,\mathrm{d}y \wedge \mathrm{d}x \\ &= (x \sin y - xy \cos y)\,\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

Math 435: Lecture 17

February 16, 2024

Reference: Spivak, pp. 90-91 and 100

Topics:

- Pulling back n-forms on $\mathbb{R}^n$
  - Theorem 4-9: if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, then
$$f^*(h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (h \circ f)(\det f')(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n).$$
  - To prove it, fix $p \in \mathbb{R}^n$ and let $A = (a_{ij}) = f'(p)$. We then have (omitting "p" from the notation after the first equality sign):
$$\begin{aligned} f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n)_p(e_1(p), \dots, e_n(p)) &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(f_*e_1, \dots, f_*e_n) \\ &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\big(\sum_{i=1}^n a_{i1}e_i, \dots, \sum_{i=1}^n a_{in}e_i\big) \\ &= \det(a_{ij})\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n(e_1, \dots, e_n) \end{aligned}$$
- Integrating n-forms on $\mathbb{R}^n$ (Spivak p. 100)
  - Let α be a (smooth) n-form defined on an open set $U \subset \mathbb{R}^n$.
    We have $\alpha = h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ for a unique smooth function h.
    Suppose (for simplicity) that h is bounded. We define:
$$\int_U \alpha = \int_U h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n := \int_U h = \int_U h(x^1, \dots, x^n)\,\mathrm{d}x^1 \dots \mathrm{d}x^n$$
  - This seems completely trivial: an n form on $\mathbb{R}^n$ is essentially a function, and we define the integral of the n-form.
  - However, we have the following fundamental fact:
    If $V \subset \mathbb{R}^n$ is another open set and $g\colon V \to U$ is an orientation-preserving diffeomorphism (meaning that $\det g'(p) > 0$ for all $p \in V$), then $\int_V g^*\alpha = \int_U \alpha$. Indeed, we have (using $\det g' = |\det g'|$)
$$\begin{aligned} \int_V g^*\alpha &= \int_V g^*(h\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \\ &= \int_V (h \circ g)\det g'\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= \int_V (h \circ g)\det g' \\ &= \int_U h \\ &= \int_U \alpha. \end{aligned}$$
  - Thus the formalism of differential forms has the change of variables formula "built in".
  - The differential form α on U is thus a "geometric" object on U which is independent of the particular way U is parametrized (as long as the orientation is preserved).
- Let us write $\Omega^k(U)$ for the set of differential k-forms defined on an open set $U \subset \mathbb{R}^n$.

- Exterior derivative
  - We now generalize the operation $\mathrm{d}\colon \Omega^0(U) \to \Omega^1(U)$ to an operation $\mathrm{d}\colon \Omega^k(U) \to \Omega^{k+1}(U)$ for all k.
  - If
    $$\omega = \sum_{i_1<\dots,i_k} \omega_{i_1,\dots,i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k},$$
    then we define
    $$\begin{aligned}\mathrm{d}\omega &:= \sum_{i_1<\dots,i_k} \mathrm{d}\omega_{i_1,\dots,i_k}\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\\ &= \sum_{i_1<\dots,i_k}\sum_{j=1}^{n} \frac{\partial}{\partial x^j}\omega_{i_1,\dots,i_k}\,\mathrm{d}x^j\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k},\end{aligned}$$
  - Theorem 4-10: (1) **Exercise:** $\mathrm{d}(\omega+\eta) = \mathrm{d}\omega + \mathrm{d}\eta$
    (2) **Exercise:** If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $\mathrm{d}(\omega\wedge\eta) = \mathrm{d}\omega\wedge\eta + (-1)^k\omega\wedge\mathrm{d}\eta$.
    (This is called the "graded Leibniz rule".)
    (3) $\mathrm{d}(\mathrm{d}\omega)$.
    Briefly, $\mathrm{d}^2 = 0$. (4) If $f\colon U \to V$ is smooth and $\omega \in \Omega^k(U)$, then $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.

Exercises:

- $\mathrm{d}(\omega+\eta) = \mathrm{d}\omega + \mathrm{d}\eta$

Math 435: Lecture 18

February 21, 2024

Reference: Spivak, pp. 91-93

Topics:

- Basic properties of the exterior derivative
  - Theorem 4-10: (1) **Exercise:** $\mathrm{d}(\omega+\eta) = \mathrm{d}\omega + \mathrm{d}\eta$
    (2) **Exercise:** If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta$.
    (This is called the "graded Leibniz rule".)
    (3) $\mathrm{d}(\mathrm{d}\omega)$. Briefly, $\mathrm{d}^2 = 0$.
    (4) If $f\colon U \to V$ is smooth and $\omega \in \Omega^k(U)$, then $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.
  - For (3), we have (writing I as a shorthand for $i_1 < \ldots < i_k$, ω_I for $\omega_{i_1,\ldots,i_k}$, and $\mathrm{d}x^I$ for $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$:
    $$\mathrm{d}\,\mathrm{d}\omega = \mathrm{d}\sum_I \sum_j \frac{\partial}{\partial x^j}\omega_I \,\mathrm{d}x^j \wedge \mathrm{d}x^I = \sum_I \sum_{j,k} \frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j}\omega_I \,\mathrm{d}x^k \wedge \mathrm{d}x^j \wedge \mathrm{d}x^I.$$
    But now switching j and k reverses the sign but gives the same result, so the sum must be zero.
  - (4) is proven by induction. We have seen that it's true for 0-forms (by the chain rule). For the induction step, we note that every $(k+1)$-form is a sum of forms of the form $\omega \wedge \mathrm{d}x^i$, and we have
    $$\begin{aligned} f^*\,\mathrm{d}(\omega \wedge \mathrm{d}x^i) &= f^*(\mathrm{d}\omega \wedge \mathrm{d}x^i) \\ &= f^*\,\mathrm{d}\omega \wedge f^*\,\mathrm{d}x^i \\ &= \mathrm{d}(f^*\omega) \wedge \mathrm{d}f^i \\ &= \mathrm{d}(f^*\omega \wedge \mathrm{d}f^i) \\ &= \mathrm{d}(f^*\omega \wedge f^*\,\mathrm{d}x^i) \\ &= \mathrm{d}f^*(\omega \wedge \mathrm{d}x^i) \end{aligned}$$
- Closed and exact forms
  - A form $\omega \in \Omega^k(U)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
  - The above theorem shows that every exact form is closed.
  - We ask: is every exact form closed?
  - The answer is: *yes* for forms on $\mathbb{R}^n$, but *no* for forms on general open subsets U.
  - Example: let us write $x = r\cos\theta$ and $y = r\sin\theta$.
    Then $\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$ and $\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$.
    Solving for $\mathrm{d}\theta$, we obtain
    $$\mathrm{d}\theta = \frac{-r\sin\theta\,\mathrm{d}x}{r^2} + \frac{r\cos\theta\,\mathrm{d}y}{r^2} = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$
  - Note that $\mathrm{d}\theta$ is defined on $U = \mathbb{R}^2 - \{(0,0)\}$.
  - Despite the notation, $\mathrm{d}\theta$ is *not* exact. The point is that θ is not actually a well-defined function on all of $\mathbb{R}^2 - \{0\}$; it is defined on $U_1 = \mathbb{R}^2 - \{(x,y) \mid x \le 0\}$ (and takes values in $(0, 2\pi)$) and it is defined on $U_2 = \mathbb{R}^2 - \{(x,y) \mid x \ge 0\}$ (and takes values in $(-\pi,\pi)$). Hence on U_1 and U_2, $\mathrm{d}\theta$ *is* an exact form; but on U it is *not*.

- However, it is *closed*, since it is exact on U_1 and U_2, hence $\mathrm{d}\,\mathrm{d}\theta(p) = 0$ for each point of $U = U_1 \cup U_2$.
- To see it is not exact, we need the general theory of integration of forms, which we don't have yet. Let us sketch the argument, as a "preview" of how this theory works. Given a curve $\gamma\colon [a,b] \to \mathbb{R}^2$, we can integrate any 1-form α over γ: $\int_\gamma \alpha$. As one might guess, for an exact form $\mathrm{d}f$, we have $\int_\gamma \mathrm{d}f = f(b) - f(a)$. In particular, if $\gamma(a) = \gamma(b)$, then $\int_\gamma \mathrm{d}f = 0$.

  Now let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Let us try to compute $\int_\gamma \mathrm{d}\theta$ (even though we do not yet know what this means!); we proceed "formally". We have $x = \cos t$ and $y = \sin t$, and hence $\mathrm{d}x = -\sin t\,\mathrm{d}t$ and $y = \cos t\,\mathrm{d}t$. Thus

$$\int_\gamma \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y = \int_0^{2\pi} \sin^2 t\,\mathrm{d}t + \cos^2 \mathrm{d}t = 2\pi$$

  Since this is not zero, γ cannot be exact.
- The amazing thing about the theory of integrating differential forms is that it makes the above "symbolic" computation rigorous.

Exercises:

- Find a differential form $\alpha \in \Omega^{n-1}(\mathbb{R}^n)$ such that $\mathrm{d}\alpha = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. (You might want to start with the case $n = 1$ or $n = 2$.)

Math 435: Lecture 19

February 23, 2024

Reference: Spivak, pp. 93-95 and 109-111

Topics:

- Poincaré lemma
  - We return to the other answer to the question: every closed $\omega \in \Omega^k(\mathbb{R}^n)$ *is exact.*
  - More generally, an open set $U \subset \mathbb{R}^n$ is *star shaped* if for each $p \in U$, the line segment $\{t \cdot p \mid 0 \leq t \leq 1\}$ is contained in U.
  - Theorem 4-11 (Poincaré Lemma): If $U \subset \mathbb{R}^n$ is star-shaped, then every closed $\omega \in \Omega^k(U)$ is exact (for $k \geq 1$).
  - Here is the proof when $k = 1$.
  - Let $\omega = \sum_{i=1}^n \omega_i \, \mathrm{d}x^i$. We have $\mathrm{d}\omega = \sum_{i,j=1}^n \mathrm{D}_j \omega_i \, \mathrm{d}x^j \wedge \mathrm{d}x^i = \sum_{i<j} (\mathrm{D}_i \omega_j - \mathrm{D}_j \omega_i) \, \mathrm{d}x^i \wedge \mathrm{d}x^j$.
    Hence ω is closed if and only if $\mathrm{D}_i \omega_j = \mathrm{D}_j \omega_i$ for all $i < j$.
  - Next suppose ω *were* exact, so $\omega = \mathrm{d}f$ and $\omega_i = \mathrm{D}_i f$ for all i.
  - Then (since U is star-shaped!) $f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f(tx)) \, \mathrm{d}t = \int_0^1 \sum_{i=1}^n \mathrm{D}_i f(tx) x^i \, \mathrm{d}t = \int_0^1 \sum_{i=1}^n \omega_i(tx) x^i \, \mathrm{d}t$.
  - Hence, given an arbitrary closed ω, we define f by the above formula.
  - Then (using $\mathrm{D}_i \omega_j = \mathrm{D}_j \omega_i$), we have

$$\begin{aligned} \mathrm{D}_j f(x) &= \int_0^1 \sum_{i=1}^n [\mathrm{D}_j \omega_i(tx) t x^i + \omega_i(tx) \delta_{ij}] \, \mathrm{d}t \\ &= \int_0^1 \left[\sum_{i=1}^n \left[\mathrm{D}_i \omega_j(tx) t x^i \right] + \omega_j(tx) \right] \mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left[\omega_j(tx) t \right] \mathrm{d}t \\ &= \omega_j(x). \end{aligned}$$

- Manifolds
  - A subset $M \subset \mathbb{R}^n$ is a *(smooth) k-dimensional submanifold of* $\mathbb{R}^n$ (or just *k-manifold*) if for each $x \in M$, there is an open subset $U \subset \mathbb{R}^n$ containing x, an open subset $V \subset \mathbb{R}^n$, and a diffeomorphism $h \colon U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V \mid y^{k+1} = \cdots = y^n = 0\}.$$

    (One can also speak of $\mathcal{C}^l$-manifolds for any $l \geq 0$ by demanding that the charts h be $\mathcal{C}^l$-diffeomorphisms rather than smooth.)
    The map h is called a *flattener.*
  - Most basic example: every open subset $U \subset \mathbb{R}^n$ is an n-manifold with since id_U is a flattener. Also, any point $p \in \mathbb{R}^n$ is an n-manifold with $f(x) = x - p$ a flattener.
  - Example: the n-sphere $\mathrm{S}^n = \{x \mid \mathbb{R}^{n+1} \mid |x| = 1\}$.
    This can be checked directly or with the following theorem.
  - A point $x \in U$ of an open subset $U \subset \mathbb{R}^n$ is a *regular point* of a smooth map $g \colon U \to \mathbb{R}^p$ is $\mathrm{D}g(x) \colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective (or equivalently, if $g'(x)$ has rank p); otherwise, it is

called a *critical point*.
A point $a \in \mathbb{R}^p$ in called a *regular value* of g if x is a regular point of g for all $x \in g^{-1}(a)$; otherwise it is called a *critical value*.

- Theorem 5-1: if $U \subset \mathbb{R}^n$ is open and $g \colon U \to \mathbb{R}^p$ is smooth and 0 is a regular value of g, then $g^{-1}(0)$ is an $(n-p)$-dimensional manifold in $\mathbb{R}^n$.

Math 435: Lecture 20

February 28, 2024

Reference: Spivak, pp. 109-112

Topics:

- Manifolds
  - A subset $M \subset \mathbb{R}^n$ is a *(smooth) k-dimensional submanifold of* $\mathbb{R}^n$ (or just *k-manifold*) if for each $x \in M$, there is an open subset $U \subset \mathbb{R}^n$ containing x, an open subset $V \subset \mathbb{R}^n$, and a diffeomorphism $h\colon U \to V$ such that
  $$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V \mid y^{k+1} = \cdots = y^n = 0\}.$$
  (One can also speak of $\mathcal{C}^l$-manifolds for any $l \geq 0$ by demanding that the charts h be $\mathcal{C}^l$-diffeomorphisms rather than smooth.)
  - Some terminology: The map h is called a *flattener*.
  By restriction, we obtain an induced map $\varphi\colon \langle h^1, \dots, h^k\rangle\colon U \cap M \to W \subset \mathbb{R}^k$ where $W = \{y \in \mathbb{R}^k \mid (y, 0) \in V\}$, called a *($k$-dimensional) coordinate chart* (or just a *chart*) for M.
  The inverse, $\varphi^{-1}\colon W \to U \cap M$ is called a *parametrization* of $U \cap M$; it is smooth, since it is the composition of the smooth map h^{-1} with the embedding $W \to V$ ($y \mapsto (y, 0)$).
  A *(k-dimensional) atlas* for M is a set of coordinate charts $\{h_\alpha\colon U_\alpha \cap M \to W\}_{\alpha \in I}$ such that the sets U_α cover M.
  Thus, a subset $M \subset \mathbb{R}^n$ is a k-manifold if and only if it admits a k-dimensional atlas.
  - Most basic example: every open subset $U \subset \mathbb{R}^n$ is an n-manifold with an atlas consisting of the single chart id_U. Also, any point $p \in \mathbb{R}^n$ is an n-manifold with $f(x) = x - p$ a flattener.
  - More interesting example: the n-sphere $\mathrm{S}^n = \{x \mid \mathbb{R}^{n+1} \mid |x| = 1\}$.
  This can be checked directly or with the following theorem.
- Manifolds as zero-sets
  - A point $x \in U$ of an open subset $U \subset \mathbb{R}^n$ is a *regular point* of a smooth map $g\colon U \to \mathbb{R}^p$ is $\mathrm{D}g(x)\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective (or equivalently, if $g'(x)$ has rank p); otherwise, it is called a *critical point.*
  A point $a \in \mathbb{R}^p$ in called a *regular value* of g if x is a regular point of g for all $x \in g^{-1}(a)$; otherwise it is called a *critical value.*
  - Theorem 5-1: if $U \subset \mathbb{R}^n$ is open and $g\colon U \to \mathbb{R}^p$ is smooth and 0 is a regular value of g, then $g^{-1}(0)$ is an $(n-p)$-dimensional manifold in $\mathbb{R}^n$.
  (Compare the situation in linear algebra: if $T\colon \mathbb{R}^n \to \mathbb{R}^p$ is a surjective linear map, then its nullspace $\ker(T)$ is an $(n-p)$-dimensional subspace.)
  - The proof uses the inverse function theorem (and is, together with the next theorem, the most important application of the latter).
  Given $x \in M := g^{-1}(0)$, we must find a flattener for M on a neighbourhood of x.
  Since $\mathrm{D}g(x)\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective, we have by the rank-nullity theorem that $\ker\big(\mathrm{D}g(x)\big)$ is $n-p$-dimensional.
  By applying an invertible linear transformation to M, we can assume $x = 0$ and $\ker\big(\mathrm{D}g(x)\big) = \mathbb{R}^{n-p} \times \{0\}$ (it's easy to see that the existence of a flattener is unaffected by applying an invertible linear transformation).

Since $\mathrm{D}g(x)$ is surjective and $\mathrm{D}g(e_i) = 0$ for $i \leq n-p$, it follows that $\mathrm{D}g(x)(e_{n-p+1}), \ldots, \mathrm{D}g(x)(e_n)$ span $\mathbb{R}^p$. Now let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-p}$ be the projection onto the first $n-p$ coordinates, and define a map $\tilde{g}\colon A \to \mathbb{R}^{n-p} \times \mathbb{R}^p \cong \mathbb{R}^n$ by $\tilde{g}(y) = \big(\pi(y), g(y)\big)$.
Since $\mathrm{D}_i g^j(x) = 0$ for $i \leq n-p$, we see that $g'(x)$ is a block-diagonal matrix, with one block $(n-p) \times (n-p)$ identity matrix, and the second block having rank p, and thus invertible. It follows that $\det g'(x) \neq 0$.
Hence, by the inverse function theorem, there is a neighbourhood $U \subset \mathbb{R}^n$ of x and $V \subset \mathbb{R}^n$ of $\tilde{g}(x) = (\pi(x), 0)$ such that $\tilde{g}|_U \colon U \to V$ is a diffeomorphism.
Since $M = g^{-1}(0)$, it follows that $\tilde{g}(U \cap M) = V \cap (\mathbb{R}^{n-p} \times \{0\})$, and we are done.

- Manifolds in terms of parametrizations
  - We have just seen that a k-manifold in $\mathbb{R}^n$ can be represented as the preimage of a regular value of a map $\mathbb{R}^n \to \mathbb{R}^{n-k}$, just as a linear k-dimensional subspace can be written as the kernel of a linear surjection $\mathbb{R}^n \to \mathbb{R}^{n-k}$.
    But we can also represent a k-dimensional linear subspace as the *image* of a *injection* $\mathbb{R}^k \to \mathbb{R}^n$. Correspondingly, we can characterize manifolds in terms of parametrizations, rather than charts:
  - Theorem 5-2: a subset $M \subset \mathbb{R}^n$ is a k-manifold if and only if for each $x \in M$, there is an open neighbourhood $U \subset \mathbb{R}^n$ of x, an open $W \subset \mathbb{R}^k$, and an injective smooth map $f\colon W \to \mathbb{R}^n$ such that (i) $f(W) = M \cap U$, (ii) $\mathrm{D}f(y)\colon \mathbb{R}^p \to \mathbb{R}^n$ is injective for all $y \in W$ (equivalently, $f'(y)$ has rank p), and (iii) $f^{-1}\colon f(W) \to W$ is continuous.
  - The proof again uses the inverses function theorem.
  - If M is a k-manifold and $x \in M$, then we have a chart $\varphi\colon M \cap U \to W \subset \mathbb{R}^k$; we have seen that the parametrization φ^{-1} is smooth, and its derivatives $\mathrm{D}(\varphi^{-1})(y)$ must be injective since it is a restriction of a diffeomorphism; also, its inverse φ is certainly continuous.
  - In the other direction, let $x \in M$ and suppose we have $f\colon W \to M \cap U$ with the given properties; we want to show that there exists a chart around x.
    Fix $y \in W$ with $f(y) = x$. Again by applying an invertible linear transformation to M, we may assume that the image of $\mathrm{D}f(y)$ is $\mathbb{R}^k \times \{0\}$.
    Now define $g\colon W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $g(y, z) = f(y) + z$.
    Since $\mathrm{D}_i g^j(y) = 0$ for $i > k$, we have that $g'(y, 0)$ is a block diagonal matrix, with a $k \times k$ block of rank k (hence invertible) followed by an identity matrix. Thus $\det g'(y, z) \neq 0$.
    By the inverse function theorem, there is some neighbourhood $V_1 \subset W \times \mathbb{R}^k$ containing $(y, 0)$ and a neighbourhood $V_2 \subset U$ such that $g(V_1) = V_2$ and $g\colon V_1 \to V_2$ is a diffeomorphism.
    Let $W' = \{y \in W \mid (y, 0) \in V_1\}$, so that $W' \subset \mathbb{R}^k$ is open. Since f^{-1} is continuous, there is some open subset $V_2' \subset V_2$ such that $f(W') = M \cap V_2'$. Let $V_1' = g^{-1}(V_2')$, so that $g\colon V_1 \to V_2$ is still a diffeomorphism.
    Then $g^{-1}\colon V_2 \to V_1$ is a flattener, since $g^{-1}(M \cap V_2) = W' \times \{0\}$.
- Transition maps
  - Suppose M is a manifold and $\varphi_1\colon M \cap U_1 \xrightarrow{\sim} W_1$ and $\varphi_2\colon M \cap U_2 \xrightarrow{\sim} W_2$ are two charts, and consider the intersection $M' = M \cap U_1 \cap U_2$.
  - We have bijections $\varphi_1|_{M'}\colon M' \to \varphi_1(M') \subset V_1$ and $\varphi_2|_{M'}\colon M' \to \varphi_2(M') \subset V_2$, and hence a bijection $\varphi_{12}\colon \varphi_1(M') \to \varphi_2(M')$ given by $\varphi_2(\varphi_1^{-1}(x))$.

  - Since φ_{12} is the composition of the smooth (parametrization) map $\varphi_1^{-1}\colon W_1 \to \mathbb{R}^n$ and the map $\varphi_2 = (h^1, \ldots, h^k)$ for some smooth map (namely a flattener) $h\colon U_2 \to V \subset \mathbb{R}^n$. Its inverse is also smooth since it is just $\varphi_{21} = \varphi_1 \circ \varphi_2^{-1}$.
  - Hence φ_{12} is a diffeomorphism; it is called the *transition map* between the charts φ_1 and φ_2.

Exercises:

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Prove that its graph $\{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ is a 1-manifold.
- Spivak 5-5

5-5. Prove that a k-dimensional (vector) subspace of $\mathbf{R}^n$ is a k-dimensional manifold.

Math 435: Lecture 21

March 1, 2024

Reference: Spivak, pp. 112-113

Topics:

- Manifolds with boundary
  - The *half-space* $\mathbb{H}^k \subset \mathbb{R}^k$ is defined as $\mathbb{H}^k = \{x \in \mathbb{R}^k \mid x^k \geq 0\}$.
  - A subset $M \subset \mathbb{R}^n$ is a *k-dimensional manifold-with-boundary* if for each $x \in M$, there exists an open neighbourhood $U \subset \mathbb{R}^n$ of x, an open set $V \subset \mathbb{R}^n$, and a smooth map $h\colon U \to V$ such that

    $$h(U \cap M) = V \cap (\mathbb{H}^k \times \{0\}) = \{y \in V \mid y^k \geq 0 \text{ and } y^{k+1} = \cdots = y^n = 0\}.$$

    We call such a map h a *flattener with boundary* (or just a *flattener*).
  - Note that if $h\colon U \to V \subset \mathbb{R}^n$ is a flattener (without boundary), then by shrinking U and adding a constant to h, we can ensure that $h^k > 0$, hence that h is flattener with boundary.

    Hence every manifold is a manifold-with-boundary.
  - Again, a half-flattener h induces a *chart* $(\varphi^1, \dots, \varphi^k)\colon U \cap M \to W$, where $W = \{y \in \mathbb{H}^k \mid (y, 0) \in V\}$. The inverse $\varphi^{-1}\colon W \to U \cap M$ is called a *parametrization*, and is a smooth map (in the sense that it extends to a smooth map $W' \to \mathbb{R}^n$ on an open subset $W' \subset \mathbb{R}^k$ containing $W \subset \mathbb{H}^k$).
  - Again, given two charts $\varphi_i\colon U_i \cap M \to W_i \subset \mathbb{H}^k$ $(i = 1, 2)$, we have a smooth transition map $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2 \cap M) \to \varphi_2(U_1 \cap U_2 \cap M)$.
  - The *boundary* $\partial M \subset M$ of a manifold-with-boundary M consists of those points $x \in M$ such that $\varphi^k(x) = 0$ for some chart $h\colon U \cap M \to V \subset \mathbb{H}^k$, we have $h^k(x) = 0$.

    WARNING: this is not to be confused with the boundary of a general subset of $\mathbb{R}^n$, as defined previously (though they may sometimes agree).
  - If $x \in \partial M$, it follows that in fact $\varphi^k(x) = 0$ for *every* chart φ. Indeed, suppose we had charts φ_1, φ_2 with $\varphi_1^k(x) = 0$ but $\varphi_2^k(x) > 0$. Then taking a small open neighbourhood $B \subset \mathrm{H}^k$ of $\varphi_1(x)$, we can assume that $y^k > 0$ for all $y \in B$. But the image $\varphi_2(\varphi_1^{-1}(B))$ of B under the transition map would be a subset of H^k containing $\varphi_2(x) \in \mathbb{R}^{k-1} \times \{0\}$, which therefore cannot be open, contradicting that the transition map is a diffeomorphism.
- Smooth maps
  - If $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are smooth manifolds with boundary, we say that a map $f\colon M \to N$ is *smooth at* $x \in M$ if there is a neighbourhood $U \subset \mathbb{R}^n$ of x such that $f\colon M \cap U \to \mathbb{R}^n$ extends to a smooth map $U \to \mathbb{R}^n$. We say that f is smooth if it is smooth at every $x \in M$.

    This is equivalent to f extending to a smooth function $U \to \mathbb{R}^n$ on an open subset $U \subset \mathbb{R}^n$ containing M.

    Indeed, if f extends to a smooth function f_x on an open neighbourhood U_x of each $x \in M$, then using a partition of unity $\{\varphi_x\}_{x \in M}$ subordinate to the cover $\{U_x\}_{x \in M}$, we have that $\sum_x \varphi_x f_x$ is a smooth extension of f to $\bigcup_{x \in M} U_x$.
  - Proposition: the following are equivalent:

    (i) f is smooth

(ii) For each $x \in M$, there are charts $\varphi_1 \colon M \cap U_1 \to W_1$ of M and $\varphi_2 \colon N \cap U_2 \to W_2$ of N such that $\varphi_2 \circ f \circ \varphi_1^{-1} \colon W_1 \to W_2$ is smooth.
(iii) For *every* chart $\varphi_1 \colon M \cap U_1 \to W_1$ of M and $\varphi_2 \colon N \cap U_2 \to W_2$ of N, the map $\varphi_2 \circ f \circ \varphi_1^{-1} \colon W_1 \to W_2$ is smooth.

- The implications (i) $\implies$ (iii) $\implies$ (ii) are straightforward.
- For the implication (ii) $\implies$ (i): given $x \in M$, consider charts $\varphi_1 \colon M \cap U_i \to W_i \subset \mathbb{R}^k$ for M and N, respectively, with $x \in U_1$ and $f(x) \in U_2$, and we consider a flattener $h_1 \colon U_1 \to V_1$ corresponding to φ_1. By assumption, $\varphi_2 \circ f \circ \varphi_1^{-1}$ is smooth.
  Then, writing $\pi \colon \mathbb{R}^m \to \mathbb{R}^k$ for the projection on the first k coordinates, we have that $\varphi_1 = \pi \circ h_1$, and $\varphi_1^{-1} \circ (\varphi_2 \circ f \circ \varphi_1^{-1}) \circ h_1 \colon U_1 \to \mathbb{R}^n$ is a smooth extension of f to U_1.
- A smooth map between manifolds is a *diffeomorphism* if it is a bijection whose inverse is also a smooth map.
- It follows that each chart $\varphi \colon U \cap M \to W$ is itself smooth (since if we choose the chart φ on M and idon W, then the map $\mathrm{id} \circ \varphi \circ \varphi^{-1} = \mathrm{id}_W$ is obviously smooth) and hence a diffeomorphism, since we already know that its inverse (the corresponding parametrization) is smooth.

Math 435: Lecture 22

March 4, 2024

Reference: Spivak, pp. 115-117

Topics:

- Tangent spaces
  - For a k-manifold $M \subset \mathbb{R}^n$ and point $x \in M$, we define a *tangent vector to M at x* to be any vector $v \in \mathbb{R}^n_x$ of the form $v = \gamma'(0)_x \in \mathbb{R}^n_x$ for a smooth curve $\gamma\colon (a,b) \to M$ (i.e., a smooth curve $\gamma\colon (a,b) \to \mathbb{R}^n$ lying entirely inside M) with $\gamma(0) = x$.
  - The set of all tangent vectors to M at x is called the *tangent space to M at x*, denoted $\mathrm{T}_x M$ (Spivak denotes it by M_x).
  - It is a linear subspace of $\mathbb{R}^n_x$ of dimension k. Indeed, if $f\colon W \to U \cap M$ is a parametrization around x, every curve on M through x is (possibly after restricting its domain) of the form $f \circ \gamma$ for a smooth curve γ in $W \subset \mathbb{R}^k$ through y, where $f(y) = x$, and hence $(f \circ \gamma)'(0)_x = f_*(\gamma'(0)_y)$.
    It follows that $\mathrm{T}_x M$ is just the image of the injective linear map $f_*\colon \mathbb{R}^k_y \to \mathbb{R}^n_x$, and hence is indeed a k-dimensional linear subspace.
  - Note that if $U \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ (hence an n-manifold), then $\mathrm{T}_x U$ is just equal to $\mathbb{R}^n_x$.
  - The standard inner product on $\mathbb{R}^n_x$ induces an inner product on $\mathrm{T}_x M$.
  - For a k-manifold with boundary M and a point x in the boundary, we define $\mathrm{T}_x M$ as the image of $f_*\colon \mathbb{R}^k_y \to \mathbb{R}^n_x$ for a parametrization $f\colon W \to U \cap M$ around x (where here $W \subset \mathbb{H}^k$).
- Derivative of a smooth map
  - Given a smooth map $f\colon M \to N$ between manifolds $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ and a point $x \in M$, we define its *derivative at x* to be the linear map $f_*\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ which takes a tangent vector $\gamma'(0)_x$ at x, where $\gamma\colon (a,b) \to M$ is a smooth curve, to the tangent vector $(f \circ \gamma)'(0)_{f(x)} \in \mathrm{T}_{f(x)} N$.
  - To see that this is indeed linear (and that it is well-defined), we note that it can be described alternative as follows: first extend f to a smooth map $F\colon U \to \mathbb{R}^n$ on an open set $U \supset M$. Then $f_*\colon \mathrm{T}_x M \to \mathrm{T}_{f(x)} N$ is just the restriction of $F_*\colon \mathbb{R}^m_x \to \mathbb{R}^n_{f(x)}$.
  - If $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are open subsets (thus and m- and n-manifold, respectively), we see that the derivative specializes to the derivative in the usual sense.
  - Immediately from the definition, we have the *chain rule*: given smooth maps $L \xrightarrow{f} M \xrightarrow{g} N$ and a point $x \in L$, we have
$$(g \circ f)_* = g_* \circ f_*\colon \mathrm{T}_x L \to \mathrm{T}_{g(f(x))} N.$$
- Vector fields on manifolds
  - A *vector field* on a smooth manifold M is a function F assigning to each $x \in M$ a tangent vector $F(x) \in \mathrm{T}_x M$. We can view F as a function $M \to \mathbb{R}^n$ (since $\mathrm{T}_x M \subset \mathbb{R}^n_p \cong \mathbb{R}^n$), and we say that F is a *smooth vector field* if this is a smooth function.
  - If $f\colon M \to N$ is any diffeomorphism, we can define a vector field $f_* F$ on N, the *pushforward of F along f*, by the formula $(f_* F)(f(y)) = f_*(F(y))$.
  - In particular, we obtain a vector field $\varphi_* F$ on each chart $U \cap M \to W \subset \mathbb{R}^k$.

  - Thus, we can think of a vector field on M as a vector field specified on each chart, such that they agree with each other when we push them forward along the transition maps.
- Differential forms on manifolds
  - A *differential p-form* ω on M is a function assigning to each $x \in M$ an element of $\Lambda^p \mathrm{T}_x M$.
  - Given a parametrization $f\colon W \to U \cap M$, we can define a pulled back p-form $f^*\omega$ on W by the usual formula $(f^*\omega)(y)(v_1, \ldots, v_p) = \omega(f(y))(f_* v_1, \ldots, v_* v_p)$.
    We define ω to be *smooth* if $f^*\omega$ is smooth for each parametrization f.
  - We write $\Omega^k(M)$ for the set of smooth k-forms on M.
  - Similarly, for any smooth map $f\colon M \to N$ between manifolds and a p-form ω on N, we can define $f^*\omega$ on M by the same formula.
  - This pullback satisfies all the familiar properties (i) $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$, (ii) $f^*(\cdot\omega) = c \cdot f^*\omega$ for a scalar c, (iii) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$, (iv) $f^*(g) = f \circ g$ for a 0-form $g\colon N \to \mathbb{R}$, and also, importantly: (iv) $(g \circ f)^*\omega = f^* g^* \omega$ given smooth maps $L \xrightarrow{f} M \xrightarrow{g} N$.
    (Here, the sum and wedge product of two differential forms on a manifold is defined pointwise, just as before.)
  - Given a p-form ω on M, we define its exterior derivative $\mathrm{d}\omega$ by the condition that for each parametrization $f\colon W \to U \cap M$, we have $f^*\,\mathrm{d}\omega = \mathrm{d}(f^*\omega)$.
  - This is independent of the chosen parametrizations since if $g\colon W' \to U \cap M$ is a second parametrization then, we have the transition map $h = f \circ g^{-1}\colon W' \to W$, and
    $$g^*(\mathrm{d}\omega) = h^* f^*(\mathrm{d}\omega) = h^*\,\mathrm{d}(f^*\omega) = \mathrm{d}(h^* f^*\omega) = \mathrm{d}(g^*\omega),$$
    where in for the third equality, we are using that we already know that pullback is compatible with exterior derivative for smooth maps between open subsets of $\mathbb{R}^k$.
  - It follows from this definition that for any smooth map $f\colon M \to N$ between smooth manifolds and a differential form ω on N, we have $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$.
    Indeed, if $g\colon W \to (M \cap U)$ and $h\colon W' \to (N \cap U')$ are parametrizations, then we need to check that $g^*(f^*\,\mathrm{d}\omega) = \mathrm{d}(g^* f^*\omega)$, and we have:
    $$\begin{aligned} g^*(f^*\,\mathrm{d}\omega) &= (h^{-1} \circ f \circ g)^*(h^*\,\mathrm{d}\omega) = (h^{-1} \circ f \circ g)^*(\mathrm{d}(h^*\omega)) \\ &= \mathrm{d}(h^{-1} \circ f \circ g)^*(h^*\omega) = \mathrm{d}(g^*(f^*\omega)). \end{aligned}$$

Math 435: Lecture 23

March 6, 2024

Reference: Spivak, pp. 117-121

Topics:

- Orientations
  - Let M be a manifold(-with-boundary) and suppose we have chosen an orientation μ_x on the vector space $\mathrm{T}_x M$ for each $x \in M$.
  - Such a choice of orientations is called *consistent* if for each $x \in M$, there is a parametrization $f : W \xrightarrow{\sim} M \cap U$ with $f(a) = x$ for some $a \in W$ and such that $[f_*((e_1)_b), \ldots, f_*(e_k)_b] = \mu_{f(b)}$ for every $b \in U$.
  - A consistent choice of orientations for M is called an *orientation* for M, and if one exists, we say that M is *orientable.* A manifold together with an orientation is called an *oriented manifold.*
  - The Möbius strip is an example of a non-orientable manifold. It can be defined explicitly as the set $\{(R + a\cos(\varphi/2))(\cos\theta, \sin\theta, 0) + (0, 0, a\sin(\varphi/2)) \mid \theta \in [0, 2\pi), a \in (-l, l)\} \subset \mathbb{R}^3$ for any $0 < r < R$.
  - If $x \in \partial M$ is a point in the boundary of a k-manifold with boundary M, then $\mathrm{T}_x(\partial M)$ is a $(k-1)$-dimensional subspace of $\mathrm{T}_x M$, and hence there are two unit vectors in $\mathrm{T}_x M$ perpendicular to $\mathrm{T}_x(\partial M)$.
    One of these is distinguished by the property of being equal to $f_*(v_0)$ for some chart $\mathbb{H}^k \supset W \xrightarrow{f} M \cap U$ where $v^k < 0$. This is called the *outward unit normal* $\mathrm{n}(x)$. This property does not depend on the particular parametrization f chosen.
  - If μ is an orientation on M, we define an *induced orientation* $\partial\mu$ on ∂M as follows. Given $x \in M$, we declare $[v_1, \ldots, v_{k-1}]$ to be in $(\partial\mu)_x$ oriented basis if and only if $[\mathrm{n}(x), v_1, \ldots, v_{k-1}] \in \mu_x$.
  - Note: applying this definition to the standard orientation on $\mathbb{H}^k$, we find that the induced orientation on $\partial\mathbb{H}^k \cong \mathbb{R}^{k-1}$ is $(-1)^k$ times the standard orientation. (However, this convention will make later formulas – specifically *Stokes' theorem* – work out nicely.
  - If $M \subset \mathbb{R}^n$ is an $(n-1)$-manifold, then every orientation induces a unit normal vector $\mathrm{n}(x) \in \mathbb{R}^n_x$ perpendicular to $\mathrm{T}_x M$; and conversely, a smooth function assigning to each $x \in M$ such a normal vector determines an orientation.
  - A smooth map $f : M \to N$ between oriented manifolds (M, μ) and (N, ν) of the same dimension is *orientation-preserving* if for each $x \in M$ and each $[v_1, \ldots, v_k] \in \mu_x$, we have $[f_* v_1, \ldots, f_* v_k] \in \mu_{f(x)}$.

Math 435: Lecture 24

March 8, 2024

Reference: Spivak, pp. 122-126

Topics:

- Integration on oriented manifolds
  - Let $M \subset \mathbb{R}^n$ be an oriented k-manifold(-with-boundary) and $\omega \in \Omega^k(M)$, and suppose that for some orientation-preserving parametrization $f \colon W \xrightarrow{\sim} M \cap U$, such that ω is zero outside of U. We then define
$$\int_M \omega = \int_W f^*\omega.$$
  - This is independent of the choice of parametrization f! Indeed, given a second such $g \colon W' \xrightarrow{\sim} M \cap U'$, we have
$$\int_{W'} g^*\omega = \int_W (h \circ f)^*\omega \int_W h^*(f^*\omega)$$
where $h \colon W' \to W$ is the transition function. (Here, we are assuming that $U = U'$, which we can do by shrinking both U and U' (and hence W and W'), since ω is by assumption zero outside of $U \cap U'$.)
But we already showed that integrals over open subsets of $\mathbb{R}^n$ are invariant under pulling back along arbitrary orientation-preserving maps!
  - Now in general, if $\omega \in \Omega^k(M)$ vanishes outside of a compact set (for example, if M itself is compact), we choose a partition of unity Φ for M subordinate to some (finite) atlas for M, so that $\int_M \varphi \cdot \omega$ is defined for all $\varphi \in \Phi$, and we define
$$\int_M \omega = \sum_{\varphi \in \Phi} \int_M \varphi \cdot \omega$$
  - It follows from the definition that if $\dim M = \dim N$ and $f \colon M \to N$ is any orientation-preserving diffeomorphism and $\omega \in \Omega^{\dim N}(N)$, then $\int_M f^*\omega = \int_N \omega$.
  - Stokes' Theorem (Theorem 5-5): If M is a compact oriented k-manifold-with-boundary and ω is a $(k-1)$-form on M, then
$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$
Here, we are considering $\omega \in \Omega^k(M)$ as a $(k-1)$-form on ∂M by restricting it: since $\mathrm{T}_x(\partial M)$ is a subspace of $\mathrm{T}_x M$ for every $x \in \partial M$, the alternating k-tensor $\omega_p \in \Lambda^k(\mathrm{T}_p M)$ also defines an alternating k-tensor on $\mathrm{T}_p(\partial M)$.
Another way to say it is that we are taking the pullback $i^*\omega$, where $i \colon \partial M \hookrightarrow M$ is the inclusion map.
  - For the proof, we first consider the case in which $M = \mathbb{H}^k$, hence $\partial M = \mathbb{R}^{k-1} \times \{0\} \subset \mathbb{H}^k$, and ω is compactly supported, meaning that it is equal to zero outside of a compact set.
  - We have
$$\omega = \sum_{i=1}^k \omega_i \, \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^i} \wedge \ldots \wedge \mathrm{d}x^k,$$

where the hat indicates omission, and hence

$$d\omega = \sum_{i=1}^{k} (-1)^{i-1} D_i\omega_i \, dx^1 \wedge \ldots \wedge dx^k$$

and hence

$$\begin{aligned}
\int_{\mathbb{H}^k} d\omega_k &= \sum_{i=1}^{k} (-1)^{i-1} \int_{\mathbb{H}^k} D_i\omega_i \\
&= (-1)^k \int_{\mathbb{R}^{k-1}} \Big(\int_0^\infty D_k\omega_i \, dx^k \Big) \, dx^1 \ldots dx^{k-1} + \\
&\quad + \sum_{i=1}^{k-1} (-1)^{i-1} \int_{\mathbb{H}^{k-1}} \Big(\int_{-\infty}^{\infty} D_i\omega_i \, dx^i \Big) \, dx^1 \ldots \widehat{dx^i} \ldots dx^k \\
&= (-1)^k \int_{\mathbb{R}^{k-1}} \omega(x_1, \ldots, x_{k-1}, 0) \, dx^1 \ldots dx^{k-1} \\
&= \int_{\partial\mathbb{H}^k} \omega.
\end{aligned}$$

where in the penultimate line we used the fundamental theorem of calculus and the fact that $\omega = 0$ outside of a bounded set, and on the last line, we use that the induced orientation on $\partial\mathbb{H}^k \cong \mathbb{R}^{k-1}$ is $(-1)^k$ times the standard orientation.

– We now consider a general $M \subset \mathbb{R}^n$. Choose a finite atlas $\{f : W_i \to U_i \cap M\}_{i=1}^N$ (this is possible by compactness of M) and a partition of unity $\Phi = \{\varphi_i\}_{i=1}^N$ subordinated to the cover $M \subset \bigcup_{i=1}^N U_i$.

We then have $\omega = \sum_{i=1}^N \varphi_i\omega$ and thus

$$\int_{\partial M} \omega = \sum_{i=1}^{N} \int_{\partial M} \varphi_i\omega$$

and

$$\int_M d\omega = \int_M d\Big(\sum_{i=1}^{N} \varphi_i\omega \Big) = \sum_{i=1}^{N} \int_M d(\varphi_i\omega).$$

Thus, we see that if we can prove the theorem for each $\varphi_i\omega$, we will also prove it for ω.

But now note that for each i, the diffeomorphism $f_i : W_i \to U_i \cap M$ restricts to a diffeomorphism $f_i : W_i \cap (\mathbb{R}^{k-1} \times \{0\}) \to (U_i \cap \partial M)$, and, writing $\eta = \varphi_i\omega$, we have

$$\int_M d\eta = \int_{U\cap M} d\eta = \int_W f^* \, d\eta = \int_W d(f^*\eta) = \int_{W\cap(\mathbb{R}^{k-1}\times\{0\})} f^*\eta = \int_{U\cap(\partial M)} \eta = \int_{\partial M} \eta.$$

Exercise:

Let $S^1 \subset \mathbb{R}^2$ be the unit circle and let α be the 1-form $\alpha = x \, dy$. Compute $\int_{S^1} x \, dy$ in two ways:

- Using Stokes' theorem
- Directly using the parametrization $f : [0, 2\pi] \to S^1$ given by $f(x) = (\cos x, \sin x)$.

Math 435: Lecture 25

March 18, 2024

Reference: Spivak, pp. 126-130

Topics:

- The volume element
  - We now want to consider a few important special cases of Stokes' theorem. For this, we will need a particular special differential form called the *volume form.*
  - If $M \subset \mathbb{R}^n$ is an oriented k-manifold, then each $\mathrm{T}_x M$ has an inner product and orientation, and hence a volume element $\omega(x) \in \Lambda^k(\mathrm{T}_x M)$.
  - This thus defines a k-form $\omega \in \Omega^k(M)$, the *volume form* or *volume element*, denoted $\mathrm{d}V$ (even though it is not necessarily the exterior derivative of anything).
  - Example: if $M \subset \mathbb{R}^3$ is an oriented surface and $n(x) \in \mathrm{T}_x M$ is the outward unit normal, then $\det(n(x), u, v) = 1$ for any oriented orthonormal basis u, v of $\mathrm{T}_x M$. Hence the volume form is $\omega(u, v) = \det(n(x), u, v) = \langle n(x), u \times v \rangle$.
  - More generally, if $M \subset \mathbb{R}^n$ is an oriented $(n-1)$-manifold (i.e., hypersurface), then $\omega(v_1, \ldots, v_{n-1}) = \det(n(x), v_1, \ldots, v_{n-1})$.
  - The *volume* of M is defined to be $\operatorname{vol}(M) = \int_M \mathrm{d}V$, and we can also define the integral of a function $f \colon M \to \mathbb{R}$ as $\int_M f \,\mathrm{d}V$.
  - A remark on this notion of volume:
    When we apply the volume element $\omega(x)$ to k tangent vectors $(v_1)_p, \ldots, (v_k)_p$, the result is the k-dimensional volume of the parallelepiped P spanned by $v_1, \ldots, v_k$.
    This volume can be defined, for example, by applying an orthogonal transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ so that P lies in $\mathbb{R}^k \times \{0\}$, and then taking the volume $|\det(Tv_1, \ldots, Tv_k)|$ of the resulting parallelepiped in $\mathbb{R}^k$.
    Equivalently, we can choose an orthonormal basis $u_1, \ldots, u_k$ for $\operatorname{Span}(v_1, \ldots, v_k)$, write $v_i = \sum_{i=1}^k a_{ij} u_j$, and then $\operatorname{vol}(P) = \det(a_{ij})$.
    Most directly, if A is the matrix with columns $v_1, \ldots, v_k$, then $\operatorname{vol}(P) = \sqrt{\det(A^\top A)}$.
    If $T \colon \mathbb{R}^k \to \mathbb{R}^n$ is an injective linear transformation with matrix A, then $\sqrt{\det(A^\top A)}$ is thus the volume of the parallelepiped spanned by $Te_1, \ldots, Te_k$, i.e., the volume-scaling factor of T (since $\operatorname{vol}(e_1, \ldots, e_k) = 1$). It is thus the direct generalization of the volume scaling factor $\det(A) = \sqrt{\det(A^\top A)}$ for a linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ with matrix A.
  - We now note that if $f \colon W \xrightarrow{\sim} U \cap M$ is a parametrization, then $\operatorname{vol}(U \cap M) = \int_W f^*\omega = \int_W \sqrt{\det((f')^\top f')}$, in direct analogy to the fact that for a diffeomorphism $f \colon V_1 \to V_2$ between open subsets of $\mathbb{R}^n$, we have $\operatorname{vol}(V_2) = \int_{V_1} |\det f'| = \int_{V_1} \sqrt{\det((f')^\top f')}$.
  - There are other equivalent characterizations of the volume as well; for instance, for a smooth 1-manifold with boundary given by a smooth curve $\gamma \colon [a, b] \to \mathbb{R}^n$, its volume is given by $\lim_{N \to \infty} \sum_{i=1}^N \|\gamma(a + (i-1)/N), \gamma(a + i/N)\|$.
- We now look at the various classical integration theorems of which Stokes' theorem is a common generalization.
- Green's Theorem

- Theorem 5-7: if $M \subset \mathbb{R}^2$ is a compact 2-manifold-with-boundary, then
$$\int_{\partial M} f\,\mathrm{d}x + g\,\mathrm{d}y = \int_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$
In order to state this theorem without reference to differential forms: if γ is a parametrization of ∂M (more precisely: ∂M is the image of a smooth map $\gamma\colon [a,b] \to \mathbb{R}^2$ with $\gamma(a) = \gamma(b)$ such that $\gamma|_{(a,b)}\colon (a,b) \to \partial M - \{\gamma(a)\}$ is a diffeomorphism), then the left-hand side is
$$\int_a^b \left[f(\gamma(t))(\gamma^1)'(t) + g(\gamma(t))(\gamma^2)'(t)\right] \mathrm{d}t.$$
- The theorem is literally just a special case of Stokes' theorem.

• Divergence theorem
- Theorem 5-8: If $M \subset \mathbb{R}^k$ is a compact k-dimensional manifold with boundary and n is the outward unit normal on ∂M, and F is a vector field on M, then
$$\int_M \mathrm{div} F = \int_{\partial M} \langle F, n\rangle\,\mathrm{d}V$$
- Most classically, this theorem is considered in the case $n = 3$.
- For the proof, let $\omega = \sum_{i=1}^k (-1)^{i-1} F^i\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k$ (where the hat denotes omission). Then $\mathrm{d}\omega = \mathrm{div} F\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$.
- We now claim that $\omega = \langle F, n\rangle\,\mathrm{d}V \in \Omega^{k-1}(\partial M)$, whence the theorem follows from Stokes' theorem. (Warning: this equality is not true if these are considered as forms on M; i.e., they only agree when applied to tangent vectors to
$partial M$.)
More precisely, we claim that for any $v_1, \ldots, v_{k-1} \in \mathrm{T}_x(\partial M)$, we have

(2) $$n^i(x)\,\mathrm{d}V(v_1, \ldots, v_{k-1}) = (-1)^{-i-1}(\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^k)(v_1, \ldots, v_{k-1}),$$

from which it follows that $\langle F, n\rangle\,\mathrm{d}V = \sum_{i=1}^k F^i n^i\,\mathrm{d}V = \omega$.
- To prove this claim, it suffices to prove it on any basis $v_1, \ldots, v_{k-1}$ of $\mathrm{T}_x(\partial M)$, which we may in particular take to be an oriented orthonormal basis, so that $n(x), v_1, \ldots, v_{k-1}$ is an orthonormal basis of $\mathbb{R}^k_x$.
- We may then write $e_i = \sum_{j=1}^{k-1} a_j v_j + bn(x)$, and we have $b = \langle e_i, n(x)\rangle = n^i(x)$.
- Recall, from a homework problem, that $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_r}(v_1, \ldots, v_r)$, for $i_1 < \ldots < i_r$, is the determinant of the $r \times r$-minor of the matrix $(v_1, \ldots, v_r)$ obtained by extracting the rows $i_1, \ldots, i_r$. Hence, the right-hand side of (2) is
$$\begin{aligned}
\det(e_i, v_1, \ldots, v_{k-1}) &= \det(\sum_j a_j v_j + bn(x), v_1, \ldots, v_{k-1}) \\
&= \det(n^i(x)n(x), v_1, \ldots, v_{k-1}) \\
&= n^i(x)\det(n(x), v_1, \ldots, v_{k-1}) \\
&= n^i(x)\,\mathrm{d}V(v_1, \ldots, v_{k-1}).
\end{aligned}$$
- QED
- The theorem shows the reason for the name "divergence". If $F(x)$ is the velocity vector at x of a moving fluid (at some time), then $\int_{\partial M}\langle F, n\rangle\,\mathrm{d}V$ is the total rate of

fluid "diverging" from M (or flowing in to M, if negative). Hence also $\mathrm{div}F(x)$ is the rate of fluid diverging from an infinitesimal ball around x.
  - An incompressible fluid one for which the amount entering any region should be the same as the amount exiting, hence this corresponds to the condition $\mathrm{div}F = 0$.
  - According to Spivak, this interpretation of div is due to Maxwell. The divergence theorem arises in Maxwell's equation as "Gauss' law", which says that the total "electric flux" through ∂M is proportional to the total amount of charge contained within M.
- (The classical) Stokes' Theorem
  - This is the original theorem after which (for some reason) the general Stokes' theorem was named.
  - Theorem 5-9: If $M \subset \mathbb{R}^3$ is a compact oriented surface with boundary and n is the outward normal, and T is the oriented unit tangent vector to ∂M (i.e., $T(x) \in \mathrm{T}_x(\partial M)$ and $\mathrm{d}V(T) = 1$), then
$$\int_M \langle \nabla \times F, n\rangle \,\mathrm{d}V = \int_{\partial M} \langle F, T\rangle \,\mathrm{d}V.$$
  - To prove it, set $\omega = F^1 \,\mathrm{d}x + F^2 \,\mathrm{d}y + F^3 \,\mathrm{d}z$.
  - Then we have $\omega = \langle F, T\rangle \,\mathrm{d}V$. Indeed, any $v \in \mathrm{T}_x(\partial M)$ is of the form $aT(x)$ for some $a \in \mathbb{R}$, and we have
$$\langle F, T\rangle \,\mathrm{d}V(aT)a\langle F, T\rangle = \langle F, aT\rangle = \omega(aT).$$
  - Next, we have
$$\begin{aligned}\mathrm{d}\omega = {} & (\mathrm{D}_1F^2 - \mathrm{D}_2F^1) \,\mathrm{d}x \wedge \mathrm{d}y + \\ & (\mathrm{D}_2F^3 - \mathrm{D}_3F^2) \,\mathrm{d}y \wedge \mathrm{d}z + \\ & (\mathrm{D}_3F^1 - \mathrm{D}_1F^3) \,\mathrm{d}z \wedge \mathrm{d}x \\ = {} & (\nabla \times F)^3 \,\mathrm{d}x \wedge \mathrm{d}y + \\ & (\nabla \times F)^1 \,\mathrm{d}y \wedge \mathrm{d}z + \\ & (\nabla \times F)^2 \,\mathrm{d}z \wedge \mathrm{d}x.\end{aligned}$$
    Hence, for $v, w \in \mathrm{T}_x(\partial M)$, we have $\mathrm{d}\omega(v, w) = (\nabla \times F) \cdot (v \times w)$.
  - We now claim, similarly, that $\langle \nabla \times F, n\rangle \,\mathrm{d}V = \mathrm{d}\omega$ (whereupon the claim of the theorem follows from Stokes' theorem). Recall that for $v, w \in \mathrm{T}_x(\partial M)$, we have $\mathrm{d}V(v, w) = \langle (v \times w), n(x)\rangle$, where $n(x)$ is the outward unit normal. But since $v, w \in \mathrm{T}_x(\partial M)$, we have $v \times w = an(x)$ for some $a \in \mathbb{R}$. Hence
$$\begin{aligned}\langle \nabla \times F, n\rangle \,\mathrm{d}V(v, w) &= \langle \nabla \times F, n\rangle\langle v \times w, n\rangle \\ &= \langle \nabla \times F, a^{-1}v \times w\rangle\langle an, n\rangle = \langle \nabla \times F, v \times w\rangle.\end{aligned}$$
  - QED
  - Again, this explains the name "curl": if F is the velocity field of a fluid, then by the theorem, $\langle \nabla \times F, n\rangle(x)$ measures the total rate at which the fluid spins around a small disc centred at x with normal vector n. If $\nabla \times F = 0$, we thus say that F is *irrotational*.
  - Again, this arises in Maxwell's equation, this time as *Ampère's law*, which says that the integral of the magnetic field generated by a constant current around any loop is proportional to the total amount of current passing through any surface spanned by the loop.

Math 435: Lecture 26

March 20, 2024

Reference: Tapp, pp. 2-4

Topics:

- We now begin the second part of the course, where instead of studying general manifolds, we will specifically study curves and surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.
  We now switch to using Tapp's book.
  This will involve a change of notation in some cases, which will take some getting used to (but that's really a good thing, since much of Spivak's notation is pretty unusual and inconvenient).
  Also, some of the stuff in Tapp's book overlaps with Spivak, so we will skip those parts.
- Parametrized curves in $\mathbb{R}^n$
  - A *parametrized curve* in $\mathbb{R}^n$ is a smooth map $\gamma\colon I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an (open, closed, or half-closed, possibly infinite) interval (i.e., a connected 1-manifold-with-boundary $I \subset \mathbb{R}$).
    We often think of the points $t \in I$ as points in *time*, and thus of $\gamma(t)$ as a moving particle.
  - γ is called a *plane curve* if $n = 2$ and a *space curve* if $n = 3$.
  - Examples:
    a circle $\gamma(t) = (\cos t, \sin t)$
    a helix $\gamma(t) = (\cos t, \sin t, t)$
    a graph $\gamma(t) = (t, f(t))$
    a line $\gamma(t) = p + t\vec{v}$.
  - Example: a graph $\gamma(t) = (t, f(t))$.
  - We write $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ for the derivative of γ.
    We can think of $\gamma'(t)$ as the *velocity* of the moving particle γ at time t.
    Similarly, the second derivative γ'' is called the *acceleration*.
  - An aside on notation: in the textbook, you'll find that Tapp writes the components of γ as $\gamma(t) = (x_1(t), \ldots, x_n(t))$. This notation reflects two useful notational conventions, which we will probably be using again.
    The first we have already discussed: on $\mathbb{R}^n$, we use $x_1, \ldots, x_n$ not only to denote the *coordinates of a given point* (which are real numbers), but also the *coordinate functions* $x_i\colon \mathbb{R}^n \to \mathbb{R}$.
    The second convention, is that, given a function $f\colon X \to Y$ between two sets and a function $g\colon Y \to Z$, it is often convenient to treat g as a function on X and simply write $g\colon X \to Z$ as a shorthand for $g \circ f\colon X \to Z$.
    In the case at hand, we have the curve $\gamma\colon I \to \mathbb{R}^n$ and the functions $x_i\colon \mathbb{R}^n \to \mathbb{R}$. According to this convention, then, we simply write $x_i\colon I \to \mathbb{R}$ for the function $x_i \circ \gamma\colon \mathbb{R}^n \to \mathbb{R}$. Thus $x_i(t)$ is "the i-th coordinate of the curve γ at time t".
  - (Note that we have switched from Spivak's notation x^i for coordinates to the more usual x_i – though in Spivak's defense, his notation is consistent with a common and very convenient general notation for tensors.)
  - The *speed* of γ at time t is $|\gamma'(t)|$.

  - The *arc length* of γ between times t_1, t_2 is $\int_{t_1}^{t_2} |\gamma'(t)|\,\mathrm{d}t$ (note: this is negative if $t_2 < t_1$). (As you will see in the homework, the arc length of a curve is the same as its volume as a 1-manifold, as defined in Spivak.)
  - γ is *regular* if $\gamma'(t)$ is non-zero for all t, and is *unit-speed* or *parametrized by arc-length* if $|\gamma'(t)| = 1$ for all t.
    The latter name comes from the fact that in this case, up to a constant, t is precisely the arc-length of γ between some fixed time t_0 and t: $\int_{t_0}^{t} |\gamma(t)|\,\mathrm{d}t = \int_{t_0}^{t} 1 = t - t_0$.
- Regular curves and 1-manifolds
  - A regular curve, intuitively, is one that has a well-defined tangent line (more specifically, well-defined tangent *direction*) at each point.
    The example $\gamma(t) = (t^3, t^2)$ illustrates this.
  - For a more extreme illustration, let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(t) = 0$ for $t \leq 0$ and f strictly increasing for $t \geq 0$ (we saw that such functions exist when discussing bump functions).
    Then for any two curves $\gamma_1, \gamma_2\colon \mathbb{R} \to \mathbb{R}^n$ with $\gamma_1(0) = \gamma_2(0) = 0$ (for example $\gamma_1(t) = (t, 0)$ and $\gamma_2(t) = (0, t)$), consider the smooth curve
$$\gamma(t) = \begin{cases} \gamma_1(f(-t)) & t \leq 0 \\ \gamma_2(f(t)) & t \geq 0. \end{cases}$$
  - These examples show that the image of an irregular curve needn't be a 1-manifold.
  - However, on the positive side, if $\gamma\colon I \to \mathbb{R}^n$ *is* regular, then each $t \in I$ has an open neighbourhood $U \subset I$ such that $\gamma(U) \subset \mathbb{R}^n$ is a 1-manifold(-with-boundary).
  - The more general fact is: if $U \subset \mathbb{R}^k$ is open and $f\colon U \to \mathbb{R}^n$ is smooth and $\mathrm{D}f(p)$ is injective for all $p \in U$, then each $p \in U$ has a neighbourhood $V \subset U$ such that $f(V) \subset \mathbb{R}^n$ is a k-manifold (and $f\colon V \to f(V)$ is a diffeomorphism).
  - This is Proposition 3.29 in Tapp, and was Theorem 5-2 in Spivak. (Though we didn't discuss it in class, it is in the lecture notes, in Lecture 20.) The proof uses the inverse function theorem, and it is a sort of "dual" to the theorem saying that a level set $f^{-1}(c)$ of a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^p$ with *surjective* derivative $\mathrm{D}f(p)$ for all $p \in f^{-1}(c)$ (i.e., c is a regular value) is a manifold.
  - Note that for a regular curve $\gamma\colon I \to \mathbb{R}^n$, even though each point $t \in I$ has a neighbourhood whose image is a manifold, it needn't be the case that the whole image $\gamma(I)$ is a manifold. An example is the curve $\gamma(t) = (t^3 - t, t^2 - 1)$.
  - Conversely, we know that if $C \subset \mathbb{R}^n$ is a 1-manifold, then there is a parametrization of C near each point; i..e, for each $p \in C$, there is an open set $U \subset \mathbb{R}^n$ and a regular curve $\gamma\colon I \to \mathbb{R}^n$ whose image is $C \cap U$.
- Eschewing Spivak's tangent space convention
  - While we're at it, let us mention some changes of notation.
  - We will drop Spivak's convention that $\mathbb{R}^n_p = \mathrm{T}_p\mathbb{R}^n$ is a different set for each $p \in \mathbb{R}^n$. From now on, we simply set $\mathrm{T}_p\mathbb{R}^n = \mathbb{R}^n$. Thus, for $\vec{v} \in \mathbb{R}^n$, we can simply write $\vec{v} \in \mathrm{T}_p\mathbb{R}^n$ instead of $\vec{v}_p \in \mathrm{T}_p\mathbb{R}^n$.
    Hence, similarly, for a k-manifold $M \subset \mathbb{R}^n$ and $p \in M$, we have $\mathrm{T}_pM \subset \mathbb{R}^n$.
  - Also, note that Tapp's notation for the derivative at $p \in M$ of a smooth map $f\colon M \to N$ is $\mathrm{d}f_p\colon \mathrm{T}_pM \to \mathrm{T}_{f(p)}N$, rather than $\mathrm{D}f(p)$.
    Other common notations are D_pf and d_pf.
    They are all fine; maybe I'll try to use Tapp's now that we are following that book.

Exercises

- At least one person from each group should get out their laptop and go to `https://www.geogebra.org/calculator`.
  It is a great tool and a lot of fun, and will come in handy for this part of the course.
- Using it, plot the circle and the helix. After that, try:
- Tapp 1.12

  EXERCISE 1.12. Use a computer graphing application to plot the following plane curves (all with domain $[0, 2\pi]$):

  (1) The **lemniscate of Bernoulli**
  $$\boldsymbol{\gamma}(t) = \left(\frac{\cos t}{1+\sin^2 t}, \frac{\sin t \cos t}{1+\sin^2 t}\right).$$
  (2) The **deltoid curve**
  $$\boldsymbol{\gamma}(t) = (2n(\cos t)(1+\cos t) - n, 2n(\sin t)(1-\cos t))$$
  for several choices of the integer $n \geq 1$.

  (3) The **astroid** curve
  $$\boldsymbol{\gamma}(t) = c\left(\cos^3 t, \sin^3 t\right)$$
  for several choices of the constant $c > 0$.

  (4) The **epitrochoid**
  $$\gamma(t) = (\cos t, \sin t) - c(\cos(nt), \sin(nt))$$
  for several choices of the integer $n > 1$ and the real number $c \in (0, 1)$.

- Tapp 1.13

  EXERCISE 1.13. Use a computer graphing application to plot the following space curves and view the plots from a variety of angles:

  (1) The **toroidal spiral**
  $$\boldsymbol{\gamma}(t) = ((4+\sin nt)\cos t, (4+\sin nt)\sin t, \cos nt)\,,\ t \in [0, 2\pi],$$
  for several choices of the positive integer n.

  (2) The **trefoil knot**
  $$\boldsymbol{\gamma}(t) = ((2+\cos 1.5t)\cos t, (2+\cos 1.5t)\sin t, \sin 1.5t)\,,\ t \in [0, 4\pi].$$
  (3) The **twisted cubic**
  $$\boldsymbol{\gamma}(t) = \left(t, t^2, t^3\right),\ t \in [-1, 1].$$

Math 435: Lecture 27

March 22, 2024

Reference: Tapp, pp. 9-13

Topics:

- Cauchy-Schwarz
  - Let V be an inner product space.
  - For $\vec{x}, \vec{y} \in V$, we have the *Cauchy-Schwarz inequality* $|\langle \vec{x}, \vec{y} \rangle| \leq |\vec{x}||\vec{y}|$, with equality iff $\vec{x} \parallel \vec{y}$ ("$\vec{x}$ and $\vec{y}$ are parallel", i.e., linearly dependent).
    To prove it: the general case is easily reduced to the case where $|\vec{x}|, |\vec{y}| = 1$.
    In this case, we want to show $|\langle \vec{x}, \vec{y} \rangle| \leq 1$. We have
    $0 \leq \|\vec{x} \pm \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}^2\| \pm 2\langle \vec{x}, \vec{y} \rangle = 2 \pm 2\langle \vec{x}, \vec{y} \rangle$, which gives $\pm\langle \vec{x}, \vec{y} \rangle \leq 1$, hence $|\langle \vec{x}, \vec{y} \rangle| \leq 1$, with equality iff $\vec{x} = \pm\vec{y}$.
  - We thus have $\frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{x}||\vec{y}|} \in [-1, 1]$ We now define the *angle* between two non-zero vectors $\vec{x}, \vec{y} \in V$ as $\angle(\vec{x}, \vec{y}) := \arccos \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{x}||\vec{y}|} \in [-1, 1] \in [0, \pi]$, so that we have the familiar formula $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos \angle(\vec{x}, \vec{y})$.
- Orthogonal projection
  - Again let V be an inner product space.
  - Given $\vec{y} \in V$ is non-zero, then any $\vec{x}$ can be uniquely decomposed as $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel} \parallel \vec{y}$ and $\vec{x}^{\perp} \perp \vec{y}$.
  - Indeed, we take $\vec{x}^{\parallel} = \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{y}|^2}\vec{y}$ and $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$.
    Uniqueness follows from the fact that if $\vec{x}^{\parallel} + \vec{x}^{\perp} = \tilde{\vec{x}}^{\parallel} + \tilde{\vec{x}}^{\perp}$ with $\vec{x}^{\parallel}, \tilde{\vec{x}}^{\parallel} \parallel \vec{y}$ and $\vec{x}^{\perp}, \tilde{\vec{x}}^{\perp} \perp \vec{y}$, then $\vec{x}^{\parallel} - \tilde{\vec{x}}^{\parallel} = \tilde{\vec{x}}^{\perp} - \vec{x}^{\perp}$ is both parallel and perpendicular to $\vec{y}$, hence zero.
  - The above formulas for $\vec{x} \mapsto \vec{x}^{\parallel}$ and $\vec{x} \mapsto \vec{x}^{\perp}$ define linear maps $\mathrm{Proj}_{\vec{y}} \colon V \to \mathrm{Span}(\vec{y})$ and $\mathrm{Proj}_{\mathrm{Span}(\vec{y})^{\perp}} \colon V \to \mathrm{Span}(\vec{y})^{\perp}$, where $\mathrm{Span}(\vec{y})^{\perp} = \{\vec{z} \in V \mid \vec{z} \perp \vec{y}\}$ is the *orthogonal complement* of $\mathrm{Span}(\vec{y})$. These are called *orthogonal projection operators* or *orthogonal projection maps.*
  - Now assume V is finite-dimensional.
  - We then, have more generally, for any linear subspace $W \subset V$, and $\vec{x} \in V$ may be uniquely decomposed as $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel} \in W$ and $\vec{x}^{\perp} \in W^{\perp} := \{\vec{z} \in W \mid \vec{z} \perp W\}$, and where $\vec{z} \perp V$ means $\vec{z} \perp \vec{w}$ for all $\vec{w} \in W$.
    Explicitly, if $\vec{y}_1, \ldots, \vec{y}_k$ is an orthonormal basis for W (which exists by the Gram-Schmidt process), then $\vec{x}^{\parallel} = \langle \vec{x}, \vec{y}_1 \rangle \vec{y}_1 + \cdots + \langle \vec{x}, \vec{y}_k \rangle \vec{y}_k$.
    We then set $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$.
    Uniqueness follows from the same argument as above.
  - Again, the formulas for $\vec{x} \mapsto \vec{x}^{\parallel}$ and $\vec{x} \mapsto \vec{x}^{\perp}$ define (linear) orthogonal projection maps $\mathrm{Proj}_W \colon V \to W$ and $\mathrm{Proj}_{W^{\perp}} \colon V \to W^{\perp}$.
- The fundamental fact about constant speed curves
  - If $\gamma \colon I \to \mathbb{R}^n$ has constant distance from the origin $|\gamma'(t)| = C$ for all $t \in I$, then $\langle \gamma, \gamma' \rangle = 0$.
  - An aside on points and vectors: as we mentioned last time, we are no longer distinguishing tangent vectors from elements in $\mathbb{R}^n$. So given a curve γ, both its position

$\gamma(t)$ at time t and its velocity vector $\gamma'(t)$ are simply elements of $\mathbb{R}^n$.
Geometrically, this is just the fact that, given a point, we have the corresponding vector from the origin to the point, and given a vector, we have the point which is the endpoint of the vector when we position it at the origin.
One should always be wary of formulas like $\langle \gamma, \gamma' \rangle = 0$ above which mix up points and vectors; there is something "ungeometric" about it, since it depends on where we situate the origin. In this particular case, the geometric interpretation of the formula is the (obvious) fact that if a curve lies on a given sphere (around *any* point!), then the curve is always tangent to the sphere (this is because the tangent space to a point on a sphere is the orthogonal complement to the vector connecting that point to the origin).

– However, there is a far more important special case of this formula: that if γ has constant *speed* $|\gamma'(t)| = C$ for all t, then $\langle \gamma', \gamma'' \rangle = 0$.
This formula is no longer suspicious since it is genuinely comparing two vectors (rather than a point and a vector).
It's meaning is: a constant speed curve is always accelerating in a direction perpendicular to its direction of motion.
– This formula follows immediately from the Leibniz rule for the inner product $\frac{\mathrm{d}}{\mathrm{d}t}\langle F(t), G(t)\rangle = \langle F'(t), G(t)\rangle + \langle F(t), G'(t)\rangle$ (which in turn follows from the usual product rule).
Indeed, if $|\gamma|$ is constant, then $0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \gamma(t)\rangle = 2\langle \gamma, \gamma' \rangle$.
(Here, we made use of another convenient abuse of notation: we may sometimes omit the argument to a function and, as we did here, just write γ instead of $\gamma(t)$.)

Exercises

- Tapp 1.8

Exercise 1.8. Compute the arc length of $\boldsymbol{\gamma}(t) = \big(2t, 3t^2\big)$, $t \in [0, 1]$.

Math 435: Lecture 28

March 25, 2024

Reference: Tapp, pp. 14-20

Topics:

- The shortest path between two points
  - Given points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, we have the curve $\gamma(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ connecting them. This curve has length $\int_0^1 |\gamma'(t)|\,\mathrm{d}t = \int_0^1 |\mathbf{q} - \mathbf{p}|\,\mathrm{d}t = |\mathbf{q} - \mathbf{p}|$.
  - Theorem: any curve γ with $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$ has arc-length $\int_a^b |\gamma'(t)|\,\mathrm{d}t \geq |\mathbf{q} - \mathbf{p}|$.
  - For the proof, write $\gamma'(t)^{\parallel}$ and $\gamma'(t)^{\perp}$ for the orthogonal decomposition with respect to the unit vector $\vec{n} = \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|}$. Then since $\gamma'(t)^{\parallel} \perp \gamma'(t)^{\perp}$, we have

    $$|\gamma'(t)|^2 = |\gamma'(t)^{\parallel} + \gamma'(t)^{\perp}|^2 = |\gamma'(t)^{\parallel}|^2 + |\gamma'(t)^{\perp}|^2 \geq |\gamma'(t)^{\parallel}|^2$$

    and hence, using that $\gamma'(t)^{\parallel} = \mathrm{Proj}_{\vec{n}}\, \gamma'(t) = \langle \gamma'(t), \vec{n} \rangle \vec{n}$:

    $$\begin{aligned}\int_a^b |\gamma'(t)|\,\mathrm{d}t &\geq \int_a^b |\gamma'(t)^{\parallel}|\,\mathrm{d}t = \int_a^b |\langle \gamma'(t), \vec{n} \rangle|\,\mathrm{d}t \geq \int_a^b \langle \gamma'(t), \vec{n} \rangle\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma(t), \vec{n} \rangle\,\mathrm{d}t = \langle \gamma(b), \vec{n} \rangle - \langle \gamma(a), \vec{n} \rangle = \langle \mathbf{q} - \mathbf{p}, \vec{n} \rangle = |\mathbf{q} - \mathbf{p}|.\end{aligned}$$
- More on acceleration
  - Given a curve γ, Tapp uses the notation $\vec{v} = \gamma'$ and $\vec{a} = \gamma''$ for its velocity and acceleration vectors.
  - Our intuition about acceleration is aided by Newton's law $\vec{F}(t) = m\vec{a}(t)$, where $\vec{F}$ is the *force* acting on the particle γ.
  - We can always decompose the acceleration orthogonally with respect to the velocity $\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}$ with $\vec{a}^{\parallel} \parallel \vec{v}$ and $\vec{a}^{\perp} \perp \vec{v}$.
  - We already observed that if γ is unit-speed then $\vec{a}^{\parallel} = 0$.
  - More generally, the size of $\vec{a}^{\parallel}$ (or more precisely, the component of $\vec{a}$ in the direction of $\vec{v}$, the absolute value of which is the size of $\vec{a}^{\parallel}$) is the rate of change of the speed of γ:
    $\frac{\mathrm{d}}{\mathrm{d}t}|\vec{v}(t)| = \frac{\langle \vec{a}, \vec{v} \rangle}{|\vec{v}|}$.
  - Indeed, $|\vec{v}| \cdot \frac{\mathrm{d}}{\mathrm{d}t}|\vec{v}| = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\vec{v}|^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \vec{v}, \vec{v} \rangle = \langle \vec{v}', \vec{v} \rangle = \langle \vec{a}, \vec{v} \rangle$.
  - The size of the perpendicular component $|\vec{a}^{\perp}|$ represents two different things: it should be larger if γ is moving faster, and also if γ is curving more sharply.
  - We will next separate out these two effects, and then concentrate on the second one: *curvature*.
- Reparametrization
  - Given a regular curve $\gamma\colon I \to \mathbb{R}^n$, a *reparametrization* of γ is a curve of the form $\tilde{\gamma} = \gamma \circ \varphi\colon \tilde{I} \to \mathbb{R}^n$ for some diffeomorphism $\varphi\colon \tilde{I} \to I$ (here, diffeomorphism just means bijection with non-vanishing derivative φ').
  - Note that any reparametrization of a regular curve is regular.
  - Also, reparametrization defines an equivalence relation on curves (i.e., this relation is *reflexive* (each curve is a reparametrization of itself), *symmetric* (each curve is a

reparametrization of each of its reparametrizations), and *transitive* (a reparametrization of a reparametrization of γ is a reparametrization of γ)).
- Any reparametrization of γ has the same image as γ.
- The image $\gamma(I) = \{\gamma(t) \mid t \in I\}$ is also called the *trace* of γ.
- A reparametrization $\tilde{\gamma} \circ \varphi$ of γ is *orientation-preserving* if φ is an orientation-preserving diffeomorphism, which here simply means $\varphi' > 0$; and otherwise it is *orientation-reversing* (so $\varphi' < 0$).

Exercises

- Tapp 1.9

EXERCISE 1.9. Let $a, b > 0$. Find the maximum and minimum speed of the ellipse $\boldsymbol{\gamma}(t) = (a \cos t, b \sin t)$.

Math 435: Lecture 29

March 27, 2024

Reference: Tapp, pp. 20-23

Topics:

- Arc-length reparametrization
  - Proposition 1.25: any regular curve $\gamma\colon I \to \mathbb{R}^n$ can be reparametrized by arc-length, i.e., there is a reparametrization of γ which is a unit-speed curve.
    To prove this, fix $t_0 \in I$ and let $s = \int_0^t |\gamma'| \, \mathrm{d}t$.
    Then s is a smooth function and $s' = |\gamma'| > 0$. It follows that $s\colon I \to s(I)$ is a diffeomorphism.
    Now consider the reparametrization $\tilde{\gamma} = \gamma \circ s^{-1}$. We have
    $$|\gamma'| = |(\tilde{\gamma} \circ s)'| = |(\tilde{\gamma}' \circ s) \cdot s'| = |\tilde{\gamma}' \circ s| \cdot |\gamma'|$$
    and hence $|\tilde{\gamma}' \circ s| = 1$, and hence $|\tilde{\gamma}'| = 1$ since s is a bijection.
    QED
  - It is very good to know that a curve can always be reparametrized by arc-length, but in practice it is usually impossible to explicitly compute the reparametrization. The reason is that this requires finding the inverse of the function $s(t) = \int_{t_0}^t |\gamma'(t)|$.
- Closed curves and periodic curves
  - A *closed curve* is a regular curve $\gamma\colon [a,b] \to \mathbb{R}^n$ such that $\gamma(a) = \gamma(b)$ and all the derivatives match: $\gamma^{(i)}(a) = \gamma^{(i)}(b)$ for all $i > 0$.
  - It is a *simple closed curve* if it is injective on $[a,b)$.
  - A curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is *T-periodic* if $\gamma(t) = \gamma(t+T)$ for all $t \in \mathbb{R}$.
  - Proposition 1.27: $\gamma\colon [a,b] \to \mathbb{R}^n$ is closed if and only if there is a $(b-a)$-periodic regular curve $\hat{\gamma}\colon \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(t) = \hat{\gamma}(t)$ for all $t \in [a,b]$.
    In one direction, we can simply define $\hat{\gamma}$ by $\hat{\gamma}(t + N(b-a)) = \gamma(t)$ for all $N \in \mathbb{Z}$, which is then regular and $(b-a)$-periodic and has the desired property.
    On the other hand, given $\hat{\gamma}$, it is clear that $\gamma\colon [a,b] \to \mathbb{R}^n$ defined by $\gamma(t) = \hat{\gamma}(t)$ is closed.
  - For *closed* curves, we make use of a slightly modified notion of reparametrization, since we also want to be able to move the "starting point". There is also the issue that a general reparametrization may no longer have matching derivatives at the endpoints, and therefore no longer be closed.
  - Thus, for $\gamma\colon [a,b] \to \mathbb{R}^n$ a closed curve, with associated $(b-a)$-periodic curve $\hat{\gamma}$, and for $\lambda \in \mathbb{R}$, we write γ_λ for the restriction of $\hat{\gamma}$ to $[a+\lambda, b+\lambda]$.
  - We then define a *closed-reparametrization* of γ (Tapp just calls this a *reparametrization*) to be a reparametrization $\tilde{\gamma} = \gamma_\lambda \circ \varphi\colon [c,d] \to \mathbb{R}^n$ of γ_λ for some $\lambda \in \mathbb{R}$, where the diffeomorphism $\varphi\colon [c,d] \to [a+\lambda, b+\lambda]$ has the additional property that its derivatives match at the endpoints: $\varphi^{(i)}(c) = \varphi^{(i)}(d)$ for all $i > 0$.
    It follows that $\tilde{\gamma}$ is again a closed curve.
  - Closed-reparametrization is again an equivalence relation.
  - We also have (Proposition 1.29): two simple closed curves have the same trace if and only if each is a closed-reparametrization of the other.

- Note that this fails for non-simply closed curves, e.g., $(\cos, \sin)\colon [0, 2\pi] \to \mathbb{R}^2$ and $(\cos, \sin)\colon [0, 4\pi] \to \mathbb{R}^2$ have the same trace.
- One direction of the proposition is obvious. For the other direction, suppose $\gamma_1\colon [a, b] \to \mathbb{R}^n$ and $\gamma_2\colon [c, d] \to \mathbb{R}^n$ have the same trace.

  By applying a closed-reparametrization to γ_2, we can assume $\gamma_2(c) = \gamma_1(a)$ and that $\langle \gamma_2'(c), \gamma_1'(a)\rangle > 0$.

  Now the restrictions $\gamma_1|_{[a,b)}$ and $\gamma_1|_{(a,b]}$ are bijections by assumption, and γ_1 is a local diffeomorphism, as is any regular curve.

  Hence we may define $\varphi\colon [c, d] \to [a, b]$ by $\varphi(u) = \gamma_1^{-1}(\gamma_2(u))$ for $u \in (c, d)$ and $\varphi(u) = a$ and $\varphi(u) = b$.

  It follows that φ is a diffeomorphism, and by definition, we have $\gamma_2 = \gamma_1 \circ \varphi$. Moreover, since γ_1, γ_2 are both closed curves, it follows that the derivatives of φ agree at the endpoints.

  Thus γ_2 is a closed-reparametrization of γ_1.

Exercises

- Tapp 1.16

  EXERCISE 1.16. Let $\boldsymbol{\gamma}$ be a **logarithmic spiral**, as defined in Exercise 1.3 on page 6. Prove that the angle between $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\gamma}'(t)$ is a constant function of t.

Math 435: Lecture 30

March 29, 2024

Reference: Tapp, pp. 24-27

Topics:

- Curvature
  - We want to quantify how sharply a (regular) curve is turning.
  - As we mentioned, this should be measured by $\vec{a}^{\perp}$, adjusted to account for the effects of velocity.
  - One way of saying this is that the curvature function $\kappa(t)$ should be independent of the parametrization; i.e., if $\tilde{\gamma} = \gamma \circ \varphi$ is a reparametrization of γ, we should have that the curvature $\tilde{\kappa}$ of $\tilde{\gamma}$ satisfies $\tilde{\kappa} = \kappa \circ \varphi$.
  - Another desideratum is simply that as the radius of a circle grows, its curvature should decrease. Specifically, we will have that a circle of radius R have curvature $\kappa(t) = 1/R$.
  - To see how to fulfill the first condition, let $\tilde{\vec{a}}$ be the acceleration of $\tilde{\gamma}$, and let us compute $\tilde{\vec{a}}^{\perp}$.

    We have
    $$\tilde{\vec{v}}(u) = \tilde{\gamma}'(u) = \varphi'(u) \cdot \gamma'(\varphi(u)) = \varphi'(u) \cdot \vec{v}\,(\varphi(u))$$
    and
    $$\tilde{\vec{a}}(u) = \tilde{\gamma}''(u) = \varphi'(u)^2 \cdot \vec{a}\,(\varphi(u)) + \varphi''(u) \cdot \vec{v}\,(\varphi(u))$$
    and hence
    $$\tilde{\vec{a}}^{\perp}(u) = \varphi'(u)^2 \cdot \vec{a}^{\,\perp}(\varphi(u)).$$
    In conclusion (using our usual abuse of notation and writing $\vec{v}$ and $\vec{a}^{\perp}$ in place of $\vec{v} \circ \varphi$ and $\vec{a}^{\perp} \circ \varphi$):
    $$\tilde{\vec{v}} = (\varphi') \cdot \vec{v} \quad \text{ and } \tilde{\vec{a}}^{\perp} = (\varphi')^2 \vec{a}^{\,\perp}.$$
  - So we see that the reparametrization scales $\vec{v}$ and $\vec{a}$ by a factor φ' and $(\varphi')^2$, respectively, so that the quantity $\frac{|\vec{a}^{\,\perp}|}{|\vec{v}\,|^2}$ is inavariant:
    $$\frac{|\vec{a}^{\,\perp}|}{|\vec{v}\,|^2} = \frac{|\tilde{\vec{a}}^{\perp}|}{|\tilde{\vec{v}}|^2}.$$
  - We therefore define the curvature of γ to be $\kappa(t) = \frac{|\vec{a}^{\,\perp}|}{|\vec{v}\,|^2}$.
  - We then have $|\vec{a}^{\,\perp}| = \kappa(t) \cdot |\vec{v}\,|^2$, in accordance with out intuition that the size of $\vec{a}^{\,\perp}$ should increase both with curvature and speed.
  - Note that for a circle $\gamma(t) = (R\cos(t), R\sin(t))$, we have $\vec{v} = (-R\sin t, R\cos t)$ and $\vec{a} = (-R\cos t, -R\sin t)$ (note that $\vec{v} \perp \vec{a}$ as it should be, since γ is unit speed; moreover, $\vec{a}$ points toward the origin, indicating that there is a "central force") and hence $\kappa(t) = |\vec{a}^{\,\perp}|/|\vec{v}\,|^2 = |\vec{a}\,|/|\vec{v}\,|^2 = R/R^2 = 1/R$, as desired.
  - Proposition 1.31: if γ is unit speed, then we simply have $\kappa(t) = |\vec{a}\,(t)|$ (since then $\vec{a} = \vec{a}^{\,\perp}$ and $|\vec{v}\,| = 1$.
  - This gives another perspective on curvature. We can imagine the tangent vector $\vec{(t)}$ of a unit speed curve as tracing out a curve on the unit-sphere, indicating at each point in time in which direction the curve is moving.

  - Then the curvature tells us how fast this "direction-curve" is moving, or in other words how fast the direction of γ is changing.
- Unit tangent and normal
  - If γ is a regular curve, we define its *unit tangent* (at time t) vector to be $\vec{t}(t) := \vec{v}(t)/|\vec{t}|$.
  - The *unit normal* vector of γ is only defined whenever $\kappa(t) \neq 0$, and is given by $\vec{n}(t) = \vec{a}^{\perp}(t)/|\vec{a}^{\perp}(t)|$.
  - By construction, $\{\vec{t}, \vec{n}\}$ is orthonormal.
  - Proposition 1.34: if γ is regular, then $\kappa(t) = |\vec{t}'|/|\vec{v}|$.
    We have
    $$\vec{a} = \vec{v}' = (|\vec{v}|\vec{t})' = |\vec{v}|'\vec{t} + |v|\vec{t}'.$$
    Since $|\vec{t}|$ is constant, we have $\vec{t} \perp \vec{t}'$, and hence $\vec{a}^{\parallel} = |\vec{v}|'\vec{t}$ and $\vec{a}^{\perp} = |\vec{v}|\vec{t}'$.
    Hence $\kappa = |\vec{a}^{\perp}|/|\vec{v}|^2 = |\vec{t}|/|\vec{v}|$.
  - Thus, can say in general, that the rate of change of the direction "per unit of speed".
  - Also note that the proof showed that $\vec{a}^{\parallel} = |\vec{v}|'\vec{t}$, as we saw before: the component of $\vec{a}$ along $\vec{v}$ is the rate of change of speed.
  - Next, the above also gave $\vec{t}' \parallel \vec{a}^{\perp}$, whence
    $$\vec{n} = \frac{\vec{a}^{\perp}}{|\vec{a}^{\perp}|} = \frac{\vec{t}'}{|\vec{t}'|}.$$
    Thee both capture the idea that $\vec{n}$ points in the direction in which γ is bending.
  - Combining the above equations gives Proposition 1.35: if γ is regular, then whenever $\kappa \neq 0$, we have $\vec{t}' = \kappa|\vec{v}|\vec{n}$, and hence
    $$\kappa|\vec{v}| = \langle \vec{t}', \vec{n} \rangle = -\langle \vec{n}', \vec{t} \rangle$$
    The last equation uses the general fact (Proposition 1.17 (2)) that if $F, G \colon I \to \mathbb{R}^n$ are orthogonal, then $\langle F', G \rangle = -\langle F, G' \rangle$, as follows immediately from the Leibniz rule for the inner product.

Math 435: Lecture 31

April 1, 2024

Reference: Tapp, pp. 28-29

Topics:

- Recap of curvature, unit tangent vector, and unit normal vector identities
  - First, the Leibniz rule: given $F, G\colon I \to \mathbb{R}^n$, we have $\langle F, G\rangle' = \langle F', G\rangle + \langle F, G'\rangle$. Hence, if $|F(t)| = 1$ for all t, then $F \perp F'$. And if $F(t) \perp G(t)$ for all t, then $\langle F', G\rangle = -\langle F, G'\rangle$.
  - Definitions (for a regular curve γ):
  $$\vec{v} = \gamma' \quad \vec{a} = \gamma'' \quad \vec{a}^{\,\parallel} = \frac{\langle \vec{a}, \vec{v}\rangle}{|\vec{v}|^2}\vec{v} \quad \vec{a}^{\,\perp} = \vec{a} - \vec{a}^{\,\parallel} \quad \vec{t} = \frac{\vec{v}}{|\vec{v}|} \quad \vec{n} = \frac{\vec{a}^{\,\perp}}{|\vec{a}^{\,\perp}|}$$
  - For a unit-speed curve γ (i.e., $|\vec{v}| = 1$):
  $$\vec{a}^{\,\parallel} = 0 \quad \vec{a}^{\,\perp} = \vec{a} \quad \vec{t} = \vec{v} \quad \vec{n} = \frac{\vec{a}}{|\vec{a}|}$$
  - Definition of curvature:
  $$\text{For a unit speed curve: } \kappa = |\vec{a}| = |\vec{a}^{\,\perp}|$$
  $$\text{In general: } \kappa = \frac{|\vec{a}^{\,\perp}|}{|\vec{v}|^2}$$
  - The unit tangent vector bends toward the unit normal:
  $$\vec{t}\,' \parallel \vec{n} \qquad \text{hence} \qquad \vec{n} = \frac{\vec{t}\,'}{|\vec{t}\,'|}$$
  - Components of the acceleration in terms of $\vec{t}$:
  $$\vec{a}^{\,\parallel} = |\vec{v}|'\vec{t} \qquad \vec{a}^{\,\perp} = |\vec{v}|\vec{t}\,' \qquad (\text{In unit speed case:} \quad \vec{a} = \vec{a}^{\,\perp} = \vec{t}\,'.)$$
  - The size of $\vec{t}\,'$:
  $$\text{For a unit speed curve: } \vec{t}\,' = \kappa\vec{n} \qquad \text{Hence: } \kappa = \langle \vec{t}\,', \vec{n}\rangle = -\langle \vec{t}, \vec{n}'\rangle$$
  $$\text{In general: } \vec{t}\,' = \kappa|\vec{v}|\vec{n} \qquad \text{Hence: } \kappa|\vec{v}| = \langle \vec{t}\,', \vec{n}\rangle = -\langle \vec{t}, \vec{n}'\rangle.$$
  - The curvature is the speed at which the direction changes:
  $$\text{For a unit speed curve: } \kappa = |\vec{t}\,'|$$
  $$\text{In general: } \kappa = \frac{|\vec{t}\,'|}{|\vec{v}|}$$
- Curvature of a graph at a critical point
  - Consider the graph $\gamma(t) = (t, f(t))$ at a critical point $f'(t_0) = 0$. We have $\vec{v} = (1, f')$ and $\vec{a} = (0, f'')$, and hence $\vec{v}(t_0) = (1, 0)$ and $\vec{a}(t_0) = (0, f''(t_0))$. These are orthogonal, so $\vec{a}^{\,\perp}(t_0) = \vec{a}(t_0)$, and hence
  $$\kappa(t_0) = \frac{|\vec{a}(t_0)|}{|\vec{v}(t_0)|^2} = |f''(t_0)|.$$

Thus, in this case, we see that the curvature is the "concavity" of f at the critical point.
(For example, if f is a parabola $f(t) = a(t-t_0)^2 + b$ centered at t_0, then $\kappa(t_0) = 2a$.)

- We can generalize this to an arbitrary point on a regular plane curve by rotating it so that $\vec{t}$ is horizontal and using the implicit function theorem. In fact, we will generalize this discussion to an arbitrary (regular) curve in $\mathbb{R}^n$.

- Review of (single-variable) Taylor polynomial
  - Theorem: for any smooth function $f\colon \mathbb{R} \to \mathbb{R}$ and any $x_0 \in \mathbb{R}$, there is a smooth function g such that
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + g(x)\cdot(x-x_0)^{n+1}.$$
  - (The firs term is called the *degree n Taylor polynomial of f*.)
  - The case $n = 1$ is called (the single-variable) *Hadamard's lemma*. The proof (in the case $x_0 = 0$, to make the notation simpler) is
$$f(x) = f(0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\big(tx\big)\,\mathrm{d}t = f(0) + x\int_0^1 f'\big(tx\big)\,\mathrm{d}t$$
and so we set $g(x) \coloneqq \int_0^1 f'(tx)\,\mathrm{d}t$, and we are done.
  - The case $n = 2$ follows from $n = 1$ by using Hadamard's lemma again: if $f(x) = f(0) + x\cdot g(x)$, then we see that $f'(0) = g(0)$, and so by Hadamard's lemma there is a smooth function h with $g(x) = g(0) + x\cdot h(x) = f'(0) + x\cdot h(x)$ and hence
$$f(x) = f(0) + x\cdot(f'(0) + x\cdot h(x)) = f(0) + x\cdot f'(0) + x^2\cdot h(x).$$
  - Then the case $n = 3$ follows by Hadamard's lemma again: we find that $\frac{f''(0)}{2} = h(0)$ and hence by Hadamard's lemma $h(x) = \frac{f''(0)}{2} + x\cdot k(x)$ and hence
$$f(x) = f(0) + x\cdot f'(0) + x^2\cdot\Big(\frac{f''(0)}{2} + x\cdot k(x)\Big) = f(0) + x\cdot f'(0) + \frac{f''(0)}{2}x^2 + k(x)\cdot x^3.$$
  - And so on. (The general case is proven by induction.)
  - For all of the elementary functions f (the trigonometric functions and their inverses, the exponential and logarithm, and all functions built out of these by arithmetic operations and composition), we have the stronger fact that f is equal to its *Taylor series* $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ (at least in some neighbourhood of x_0).
  - Functions with this property are called *analytic*.
    Warning: not every smooth function is analytic!
    For example, in our study of bump functions, we saw smooth f such that $f^{(k)}(0) = 0$ for all k and yet $f(x) > 0$ for all $x > 0$.
  - In light of Taylor's theorem, every smooth function f has a "second-order approximation" near any $x_0 \in \mathbb{R}$ which for a critical point x_0 is a parabola $a(x-x_0)^2 + b$ centered at x_0 and tangent to the graph of f. As above, the curvature of the graph of f at x_0 is then $2a$.

Exercise

- Tapp 1.38

EXERCISE 1.38. For the helix in Example 1.3, compute the curvature function:

(1) directly from Definition 1.30,
(2) by reparametrizing by arc length and using Proposition 1.31.

(The helix in question is $\gamma(t) = (\cos t, \sin t, t)$.)

Math 435: Lecture 32

April 3, 2024

Reference: Tapp, pp. 29-35

Topics:

- Osculating planes and circles
  - Assume that $\gamma\colon I \to \mathbb{R}^n$ is unit speed and fix $t_0 \in I$ with $\kappa(t_0) \neq 0$. The (component-wise) second-order Taylor polynomial of γ at t_0 is
$$\begin{aligned}\gamma(t_0+h) &\simeq \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0)\\ &= \gamma(t_0) + h\vec{t} + \frac{\kappa h^2}{2}\vec{n}\end{aligned}$$
  - The *osculating plane* of γ at t_0 is $\mathrm{Span}(\vec{t}(t_0), \vec{n}(t_0)) = \mathrm{Span}(\vec{v}(t_0), \vec{a}(t_0))$.
    (It is convenient to define it in this way as a linear subspace of $\mathbb{R}^n$, but it's more natural to think of it centered at $\gamma(t_0)$, i.e., to consider the plane $\{\gamma(t_0) + x\vec{t}(t_0) + y\vec{n}(t_0) \mid x, y \in \mathbb{R}\}$.)
  - Hence, we see that near $\gamma(t_0)$, the trace of γ is approximated by the parabola $(x, y) = (h, \frac{\kappa h^2}{2})$, i.e., $y = \kappa x^2/2$, of concavity κ in the osculating plane.
  - Next, the *osculating circle* to γ at t_0 is the circle of radius $1/\kappa(t_0)$ in the osculating plane centered at the origin. It can be parametrized as
$$\mathbf{c}(s) = \frac{1}{\kappa(t_0)}(\cos(s)\vec{t} + \sin(s)\vec{n}), \quad s \in [0, 2\pi].$$
  - Again, it is more natural to consider the circle situated elsewhere, namely centered at $\epsilon(t_0) = \gamma(t_0) + \frac{1}{\kappa(t_0)}\vec{n}$, so that it is tangent to γ at $\gamma(t_0)$.
  - Any three non-colinear points determine a circle, and the osculating circle was originally defined to be the circle passing through $\gamma(t_0)$ and two infinitesimally nearby points. A definition in this spirit can still be made using limits.
    (If the definition is made this way, then the curvature can be *defined* as the reciprocal of the radius (which is possibly ∞) of the osculating circle.)
  - The centers $\epsilon(t)$ of the osculating circles at $\gamma(t)$ themselves form a curve, called the *evolute* of γ.
- Plane curves
  - In general, we can only talk about the *size* of the curvature, as a positive number.
  - In the plane, we can also ask whether the curve is bending left or right.
  - Define $R_{90}\colon \mathbb{R}^2 \to \mathbb{R}^2$ (or just R for short) to be the rotation by 90 degrees map $R_{90}(x, y) = (-y, x)$.
  - This is also multiplication by i in the complex plane.
  - Note that for a curve γ with non-zero curvature, we have $R\vec{t} = \vec{n}$ if γ is bending left, and $R\vec{t} = -\vec{n}$ if it is bending right.
  - Recall that for a unit-speed curve γ, we have $\vec{t}\,' = \kappa\vec{n}$ and hence $\kappa = \langle \vec{t}\,', \vec{n}\rangle$.
  - We thus defined the *signed curvature* of (the unit-speed curve γ) to be $\kappa_{\mathrm{s}} = \langle \vec{t}\,', R\vec{t}\rangle$.
  - We thus have $\vec{t}\,' = \kappa_{\mathrm{s}} \cdot R\vec{t}$, and $\kappa = |\kappa_{\mathrm{s}}|$ (and specifically, $\kappa = \kappa_{\mathrm{s}}$ when γ is bending left and $\kappa = -\kappa_{\mathrm{s}}$ when γ is bending right).

- For a general regular curve γ, we have $\vec{t}\,' = \kappa|\vec{v}\,|\vec{n}$ and hence $\kappa|\vec{v}\,| = \langle\vec{t}\,', \vec{n}\rangle$, and we thus define $\kappa_{\mathrm{s}} = \frac{\langle\vec{t}\,', R\vec{t}\rangle}{|\vec{v}\,|}$.
- This quantity is invariant under *orientation-preserving* reparametrizations. Unsurprisingly, an orientation-reversing reparametrizations changes its sign.

- Signed curvature as rate of change of angle.
  - We have interpreted the curvature of a unit-speed curve as the *speed* of the change of direction of γ.
  - For a plane curve, the direction is simply given by an angle. Let us now interpret the *signed* curvature as the *rate of change of the angle* (positive if turning left, negative if turning right).
  - There is one issue with this, which is that the angle is only defined up to a multiple of 2π; so first we have to show that we can consistently assign an angle of γ' at each point along the curve.
  - Proposition 1.39: if $\gamma\colon I \to \mathbb{R}^2$ is a regular curve, then there is a smooth *angle function* $\theta\colon I \to \mathbb{R}$ with $\vec{t}\,(t) = (\cos\theta(t), \sin\theta(t))$ for all t.

Exercises

- Show that the curvature of a regular curve is constantly zero if and only if its trace lies on a line.

Math 435: Lecture 33

April 5, 2024

Reference: Tapp, pp. 35-43

Topics:

- Signed curvature as rate of change of angle.
  - Returning to Proposition 1.39: if $\gamma\colon I \to \mathbb{R}^2$ is a regular curve, then there is a smooth *angle function* $\theta\colon I \to \mathbb{R}$ with $\vec{t}(t) = (\cos\theta(t), \sin\theta(t))$ for all t.
    This function is unique up to adding an integer multiple of 2π.
    (More generally, for any $F\colon I \to \mathbb{R}^2$ with $|F| = 1$, there is an angle function θ with $F(t) = (\cos\theta(t), \sin\theta(t))$.)
  - The uniqueness is clear, since given two such functions θ_1, θ_2, we have that $\theta_1(t) - \theta_2(t)$ is a multiple of 2π for each t, and since $\theta_1 - \theta_2$ is continuous, it must be constant.
  - Before proving the existence, let us check that $\kappa_{\mathrm{s}} = \theta'$ for a unit-speed curve.
  - By definition, we have $\kappa_{\mathrm{s}} = \langle \vec{t}', R\vec{t} \rangle$.
    Since $\vec{t} = (\cos\theta, \sin\theta)$ by definition, we thus have $\vec{t}' = (-\theta' \sin\theta, \theta' \cos\theta) = \theta' \cdot R\vec{t}$ and hence $\kappa_{\mathrm{s}} = \langle \theta' \cdot R\vec{t}, R\vec{t} \rangle = \theta'$, as desired.
  - Now fix some $t_0 \in I$ and some θ_0 such that $\vec{t}(t_0) = (\cos\theta_0, \sin\theta_0)$.
  - From the above observation, we see that we have no choice but to define $\theta(t) = \int_{t_0}^{t} \kappa_{\mathrm{s}}(u)\,\mathrm{d}u$.
  - We need to check that with this definition, we indeed have $\vec{t}(t) = (\cos\theta(t), \sin\theta(t))$ for all t.
  - For this, we note that in a small neighbourhood $U \subset I$ of any $t \in I$, we *can* find a "local" angle function Θ with $\vec{t}(t) = (\cos\Theta(t), \sin\Theta(t))$ for all $t \in U$.
    This is because, by continuity of $\vec{t}$, the image any sufficiently small U will lie in $\mathrm{S}^1 - \{(1,0)\}$ or $\mathrm{S}^1 - \{(0,1)\}$, and so we can define Θ by composing $\vec{t}\colon U \to \mathrm{S}^1$ with the inverse of the diffeomorphism $(\cos, \sin)\colon (0, 2\pi) \to \mathrm{S}^1 - \{(1,0)\}$ or $(\cos, \sin)\colon (-\pi, \pi) \to \mathrm{S}^1 - \{(-1,0)\}$.
  - Now, to see that θ has the desired property, let $\widehat{I} \subset I$ be the set of points t with $\vec{t}(t) = (\cos\theta(t), \sin\theta(t))$. We want to show $\widehat{I} = I$.
  - We have $\widehat{I} \neq \emptyset$ since $t_0 \in \widehat{I}$.
  - We then show that $\widehat{I}$ is both open and closed in I, and the appeal to the general fact that an open and closed subset of a connected set must be the whole set.
  - $\widehat{I}$ is closed because the set where two continuous functions (here, $\vec{t}$ and $(\cos\theta, \sin\theta)$) agree is always closed.
  - It is open by the above observation, since near any point t, we have a local angle function Θ, and if $t \in \widehat{I}$, then we can assume that $\Theta(t) = \theta(t)$ by adding a multiple of 2π to Θ. But now $\Theta' = \kappa_{\mathrm{s}} = \theta$, and they agree at one point, hence they must be equal.
  - This ends the proof.
  - Using the angle function θ, we can define the "number of times a closed plane curve turns around".

  - Namely, if $\gamma\colon [a,b] \to \mathbb{R}^2$ is a closed plane curve, we define its *rotation index* to be $\frac{1}{2\pi}(\theta(b) - \theta(a))$, where θ is an angle function.
  - This is independent of the chosen angle function, since they always differ by a constant, and it is always an integer since we have $\vec{t}(a) = \vec{t}(b)$ and hence $\theta(a)$ and $\theta(b)$ differ by a multiple of 2π.
- Space curves; the Frenet frame
  - We now consider the curvature of curves in $\mathbb{R}^3$; here, signed curvature no longer makes sense.
  - We recall some basic properties of the cross product.
    * Definition: $\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - v_2u_3)$.
    * It is an alternating, bilinear operation.
    * When $\vec{u}, \vec{v}$ are independent, $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}, \vec{v}$, and moreover $(\vec{u}, \vec{v}, \vec{u} \times \vec{v})$ is an oriented basis (i.e., $\det(\vec{u}, \vec{v}, \vec{u} \times \vec{v}) > 0$).
    * In fact, we have $\det(\vec{u}, \vec{v}, \vec{w}) = \langle \vec{u} \times \vec{v}, \vec{w} \rangle$ for any $\vec{u}, \vec{v}, \vec{w}$ (which implies the previous fact by taking $\vec{w} = \vec{u} \times \vec{v}$).
    * We have $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ where $\theta = \angle(\vec{u}, \vec{v})$.
    * (Together with the above facts, this characterizes $\vec{u} \times \vec{v}$ uniquely.)
    * If $F, G\colon I \to \mathbb{R}^3$ are smooth, then $(F \times G)' = F' \times G + F \times G'$. (More generally, this holds for any bilinear operation.)
  - Proposition 1.46: if γ is a regular space curve, then

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}.$$

    Recalling that $\kappa = |\vec{a}^{\perp}|/|\vec{v}|^2$, this follows immediately from the fact that $|\vec{a}^{\perp}| = |\vec{a}| \cdot \sin\theta = |\vec{v} \times \vec{a}|/|\vec{v}|$.
  - Next, using the cross product, we extend our orthogonal set $(\vec{t}, \vec{n})$ to an orthogonal basis: if γ is a regular curve and $t \in I$ satisfies $\kappa(t) = 0$, we define the *Frenet frame* to be the orthonormal basis $(\vec{t}(t), \vec{n}(t), \vec{b}(t))$ of $\mathbb{R}^3$, where $\vec{b}(t) := \vec{t}(t) \times \vec{n}(t)$ is called the *unit binormal vector* of γ.
    Recall that $\vec{t} = \vec{v}/|\vec{v}|$ and $\vec{n} = \vec{a}^{\perp}/|\vec{a}^{\perp}| = |\vec{t}'||\vec{t}|'$.
  - Note that the vector $\vec{b}$ is a unit-normal vector to the osculating plane. Hence the curve $\mathbf{0} + \vec{b}(t)$ on S^2 indicates the "tilt" of the osculating plane at each point in time.
- Torsion
  - We now want to define the *torsion* as the speed at which this tilt changes, i.e., the speed of the curve $\vec{b}(t)$. As first pass, this would mean simply taking $|\vec{b}|'$ (in the unit-speed case, or $|\vec{b}|'/|\vec{v}|$ in general), but we will see that we can actually define a *signed* quantity, indicating whether $\vec{b}$ is twisting to the left or right.
  - To this end, note that $\vec{b}' \perp \vec{b}$ (since $|\vec{b}| = 1$) and $\vec{b}' \perp \vec{t}$, since $\langle \vec{b}', \vec{t} \rangle = -\vec{b}\vec{b}, \vec{t}' = 0$ (since $\langle \vec{b}, \vec{t} \rangle = 0$ and since $\vec{t}' \parallel \vec{n} \perp \vec{b}$).
  - It follows that $\vec{b}'$ is a multiple of $\vec{n}$, namely $\vec{b}' = \langle \vec{b}', \vec{n} \rangle \vec{n}$, which is positive if $\vec{b}$ is twisting toward $\vec{n}$, and negative otherwise.
  - We thus define the *torsion* of a unit-speed curve γ with non-vanishing curvature to be $\tau = -\langle \vec{b}', \vec{n} \rangle$. (Some people define torsion to have the opposite sign.)

- For a general regular curve (still with $\kappa = 0$), we define $\tau = \frac{-\langle \vec{b}\,', \vec{n} \rangle}{|\vec{v}\,|}$; one checks that τ is invariant under reparametrization (not just orientation-preserving reparametrization!).
- Thus, the torsion is positive if the Frenet frame is twisting "away" from the direction of curvature, and negative if it is twisting "towards" it.

Math 435: Lecture 34

April 8, 2024

Reference: Tapp, pp. 44-56

Topics:

- Torsion and planarity
  - Proposition 1.51: if γ is a regular space curve with non-vanishing curvature, then the trace of τ lies in a plane if and only if $\tau = 0$.
  - First suppose that γ is in the (x, y)-plane, so that $\gamma = (x(t), y(t), 0)$; hence $\vec{v} = (x', t', 0)$ and hence $\vec{v}$ and thus $\vec{t}$ lie in the (x, y)-plane. Next, $\vec{a} = (x'', y'', 0)$, hence $\vec{a}$, hence also $\vec{a}^{\perp}$ and $\vec{n}$, lie in the (x, y)-plane.

    Thus $\vec{b}$ is orthogonal to the (x, y)-plane, and hence constant by continuity. Thus $\vec{b}' = 0$ and hence $\tau = 0$.
  - The argument for a general plane P is similar. Letting $\vec{u}$ be a unit normal to the plane, we find that $\vec{t} \perp \vec{u}$ and $\vec{n} \perp \vec{u}$ and thus conclude $\vec{b} = \pm\vec{v}$ is constant and hence $\vec{b}' = 0$ and $\tau = 0$.
  - Conversely, suppose that $\tau = 0$, so that $\vec{b} = (a, b, c)$ is constant. We claim that $d \coloneqq \langle \gamma, \vec{b} \rangle$ is constant, so that γ lies in the plane $ax + by + cz = d$.

    Indeed, we have $\langle \gamma, \vec{b} \rangle' = \langle \gamma', \vec{b} \rangle + \langle \gamma, \vec{b}' \rangle = 0 + 0$.
- Third-order Taylor polynomial
  - Let $\gamma \colon I \to \mathbb{R}^3$ be a unit-speed space curve with non-zero curvature at $t_0 \in I$. Its third-order Taylor polynomial is:

    $$\gamma(t_0 + h) \simeq \gamma(t_0) + \gamma'(t_0)h + \frac{\gamma''(t_0)}{2}h^2 + \frac{\gamma'''(t_0)}{6}h^3.$$

    We have (remembering that γ is unit-speed)

    $$\gamma' = \vec{t} \qquad \gamma'' = \vec{t}' = \kappa\vec{n} \qquad \gamma''' = (\kappa\vec{n}) = \kappa'\vec{n} + \kappa\vec{n}'.$$

    (where we are writing $\vec{t}, \vec{n}, \vec{b}$ for $\vec{t}(t_0), \vec{n}(t_0), \vec{b}(t_0)$).
  - Let's compute $\vec{n}'$ (in general, for a regular (not necessarily unit-speed) curve with non-vanishing curvature). We have $\vec{n} = \vec{b} \times \vec{t}$ and hence

    $$\vec{n}' = \vec{b}' \times \vec{t} + \vec{b} \times \vec{t}' = -|\vec{v}|\tau\vec{n} \times \vec{t} + |\vec{v}|\kappa\vec{b} \times \vec{n} = -|\vec{v}|\kappa\vec{t} + |\vec{v}|\kappa\vec{b}.$$
  - Combining this with the definitions of κ and τ, we obtain

    $$\begin{array}{rcccc} \vec{t}' & = & & |\vec{v}|\kappa\vec{n} & \\ \vec{n}' & = & -|\vec{v}|\kappa\vec{t} & & +|\vec{v}|\tau\vec{b} \\ \vec{b}' & = & & -|\vec{v}|\tau\vec{n} & \end{array}$$

    or in brief:

    $$\begin{bmatrix} \vec{t} \\ vn \\ vb \end{bmatrix}' = |\vec{v}| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}.$$

    These are called the *Frenet-Serret equations*, a system of ordinary differential equations suggestively involving a skew-symmetric matrix.

- Returning to the matter at hand, we thus have
$$\gamma(t_0+h) \simeq \gamma(t_0) + h\vec{t} + \frac{h^2\kappa}{2}\vec{n} + \frac{h^3}{6}(\kappa'\vec{n} - \kappa^2\vec{t} + \kappa\tau\vec{b}).$$
Writing this in the coordinates
$$\gamma(t_0) + x\vec{t} + y\vec{n} + z\vec{b},$$
we obtain
$$\begin{array}{llll} x = & h & & -\frac{\kappa^2}{6}h^3 \\ y = & & \frac{\kappa}{2}h^2 & \frac{\kappa'}{6}h^3 \\ z = & & & \frac{}{\kappa\tau}6h^3. \end{array}$$
- In particular, since κ is always positive, we see that z is increasing if and only if $\tau > 0$.
- In conclusion: if the torsion is zero, then the curve lies *inside* the osculation plane; otherwise the *sign* of the torsion indicates whether the curve is moving upwards or downwards through the osculating plane.

• Rigid motions
  - A *rigid motion* (or *isometry*) of $\mathbb{R}^n$ is a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $|f(\mathbf{p}) - f(\mathbf{q})| = |\mathbf{p} - \mathbf{q}|$ for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$.
  - Any translation $T_{\vec{v}}(\mathbf{p}) = \mathbf{p} + \vec{v}$ with $\vec{v} \in \mathbb{R}^n$ is clearly a rigid motion.
  - Next, if $A \in \mathrm{O}(n)$ an orthogonal matrix (here, $\mathrm{O}(N) \subset \mathbb{R}^{n\times n}$ is the set of orthogonal matrices), then $L_A(\mathbf{p}) = A \cdot \mathbf{p}$ is a rigid motion.
  - Recall that A being orthogonal means that $A^\top A = I$ or equivalently that the columns of A form an orthonormal basis.
  The first condition is equivalent to $\vec{u}^{\,\top} A^\top A \vec{v} = \vec{u}^{\,\top} I \vec{v}$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$, i.e., that $\langle A\vec{u}, A\vec{v}\rangle = \langle \vec{u}, \vec{v}\rangle$, i.e., that A preserves the inner product.
  This obviously implies that $\langle A\vec{v}\rangle = |\vec{v}|$ for all $\vec{v}$, i.e., that A preserves lengths.
  But conversely, since $\langle \vec{u}, \vec{v}\rangle = \frac{1}{2}(|\vec{u} + \vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2)$, if A preserves lengths, then it preserves the inner product, hence is orthogonal.
  Finally, if A preserves lengths, then $|A\mathbf{p} - A\mathbf{q}| = |A(\mathbf{p} - \mathbf{q})| = |\mathbf{p} - \mathbf{q}|$ for any $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, i.e., A preserves distances.
  We conclude that A is in fact orthogonal *if and only if* L_A is a rigid motion.
  - Next, the composition of two rigid motions is a rigid motion. Hence $f(\mathbf{p}) = A \cdot \mathbf{p} + \vec{v}$ is a rigid motion for any orthogonal $A \in \mathbb{R}^{n\times n}$ and $\vec{v} \in \mathbb{R}^n$.
  - Conversely, Proposition 1.58: every rigid motion f is of the form $f(\mathbf{p}) = A \cdot \mathbf{p} + \vec{v}$ for a unique orthogonal A and vector $\vec{v}$.
  - To prove it, first suppose that $f(\mathbf{0}) = \mathbf{0}$.
  Then, since f preserves distances and preserves the origin, it also preserves distances form the origin, $|f(\vec{v})| = |\vec{v}|$.
  Hence, it preserves inner products by the above argument.
  It follows that the vectors $f(\vec{e}_1), \ldots, f(\vec{e}_n)$ are an orthonormal basis.
  Let A be the matrix having these vectors as columns, so that A is an orthogonal matrix and $A \cdot \vec{e}_i = f(\vec{e}_i)$.
  Consider the map $g = L_{A^{-1}} \circ f = L_A^{-1} \circ f$. Since f and A^{-1} both preserve inner products, so does g. Moreover, $g(\vec{e}_i) = L_A^{-1}(f(\vec{e}_i)) = \vec{e}_i$ for all i. Hence, for any $\vec{v}$, we have $g(\vec{v})_i = \langle g(\vec{v}), \vec{e}_i\rangle = \langle g(\vec{v}), g(\vec{e}_i)\rangle = \langle \vec{v}, \vec{e}_i\rangle = \vec{v}_i$ for all i and hence $g(\vec{v}) = \vec{v}$.
  In other words g is the identity map, so that $f(\vec{v}) = L_A$, as desired.

  - Finally, if f is an arbitrary rigid motion, and $f(\mathbf{p}) = \vec{v}$, then $g(\mathbf{p}) = f(\mathbf{p}) - \vec{v}$ is a rigid motion preserving the origin, and $g = L_A$ for some $A \in \mathrm{O}(n)$ and $f(\mathbf{p}) = g(\mathbf{p}) + \vec{v} = A \cdot \mathbf{p} + \vec{v}$.
  - A rigid motion $f = T_{\vec{v}} \cdot L_A$ is *proper* (or *orientation-preserving*) if the matrix A is orientation-preserving, i.e., $\det(A) > 0$ (i.e., $\det(A) = 1$ since $A \in \mathrm{O}(n)$), and otherwise *improper* or *orientation-reversing*.

Exercises

- Tapp 1.47

  EXERCISE 1.47. Prove that the trace of every regular plane curve with constant nonzero signed curvature must equal a circle or a segment of a circle.

Math 435: Lecture 35

April 10, 2024

Reference: Tapp, pp. 56-57

Topics:

- Rigid transformations of curves
  - Given a curve $\gamma\colon I \to \mathbb{R}^n$ and a rigid motion $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we obtain a new curve $f \circ \gamma$.
  - Proposition 1.64. The curvature of a curve in $\mathbb{R}^n$, the signed curvature of a curve in $\mathbb{R}^2$, and the torsion of a curve in $\mathbb{R}^3$ are all invariant under rigid motions. Improper rigid motions also preserve curvature, but multiply signed curvature and torsion by -1.
  - Invariance means that, for example, if f is a proper rigid motion and $\hat{\gamma} = f \circ \gamma$, then the curvatures $\kappa, \hat{\kappa}\colon I \to \mathbb{R}$ of γ and $\hat{\gamma}$ are equal.
  - (Note this has a slightly different meaning than *invariance under reparametrization*. In the latter case, if $\varphi\colon \tilde{I} \to I$ is a diffeomorphism and $\tilde{\gamma} = \gamma \circ \varphi\colon \tilde{I} \to \mathbb{R}^n$ has curvature $\tilde{\kappa}\colon I \to \mathbb{R}$, then the invariance of curvature means $\tilde{\kappa} = \kappa \circ \varphi$ – the simpler equation $\tilde{\kappa} = \kappa$ doesn't make sense since $\tilde{\kappa}$ and κ have different domains.)
  - The proof comes down to the fact that if $f = T_{\vec{v}} \circ L_A$, then $\hat{\vec{v}} = \hat{\gamma}' = (f \circ \gamma)' = (A \cdot \gamma + \vec{v}\,)' = A \cdot \gamma' = A\vec{v}\,$, and by a similar calculation, we have $\hat{\vec{u}} = A\vec{u}\,$, whenever $\vec{u}$ is one of $\vec{t}\,, \vec{n}, \vec{b}\,, \vec{t}\,', \vec{n}', \vec{b}\,'$. Since $\kappa, \kappa_{\mathrm{s}}, \tau$ are all defined as inner products between these quantities, and A preserves inner products.
  - Conversely, we have Theorem 1.65 (the "fundamental theorems of plane and space curves"):
    (1) Two unit-speed plane curves differ by a proper rigid motion if and only if they have the same signed curvature. Moreover, for *any* prescribed (smooth) functions $\kappa_s\colon I \to \mathbb{R}$, there is a (unique up to proper rigid motions) unit-speed plane curve $\gamma\colon I \to \mathbb{R}^2$ with signed curvature κ_s.
    (ii) Two space curves with non-vanishing curvature differ by a proper rigid motion if and only if they have the same curvature and torsion. Moreover, for *any* prescribed (smooth) functions $\kappa\colon I \to \mathbb{R}_{>0}$ and $\tau\colon I \to \mathbb{R}$, there is a (unique up to proper rigid motions) unit-speed space curve $\gamma\colon I \to \mathbb{R}^3$ with curvature κ and torsion τ.
  - Part (i) follows from the fundamental theorem of calculus. Given the function κ_{s}, fix $t_0 \in I$ define $\theta(t) = \int_{t_0}^t \kappa_s(t)\,\mathrm{d}t$ and $\vec{v}\,(t) = (\cos\theta(t), \sin\theta(t))$ and $\gamma(t) = \int_0^t \vec{v}\,(t)\,\mathrm{d}t$. Then γ is a plane curve with velocity $\gamma' = \vec{v}$ (hence unit speed) and angle function θ and hence signed curvature $\kappa_{\mathrm{s}} = \theta'$.
    For the uniqueness, suppose γ and $\hat{\gamma}$ are unit speed with equal signed curvature $\kappa_{\mathrm{s}} = \hat{\kappa}_{\mathrm{s}}$. Fix $t_0 \in I$. By applying a rigid motion, we can assume that $\gamma(t_0) = \hat{\gamma}(t_0)$ and $\gamma'(t_0) = \hat{\gamma}'(t_0)$.
    In particular, by possibly shifting one of them by a multiple of 2π, we can ensure that their angle functions agree at t_0: $\theta(t_0) = \hat{\theta}(t_0)$.
    But then since $\theta' = \kappa_{\mathrm{s}} = \hat{\kappa}_{\mathrm{s}} = \hat{\theta}'$, it follows that $\theta = \hat{\theta}$.
    But then since $\gamma = (\cos\theta, \sin\theta) = (\cos\hat{\theta}, \sin\hat{\theta}) = \hat{\gamma}'$ and $\gamma(t_0) = \gamma'(t_0)$, it follows that $\gamma = \hat{\gamma}$.
    (This may seems to be proving too much! We wanted to show that $\gamma, \hat{\gamma}$ at most *differ*

by a rigid motion, but we showed that they're *equal*! The point is that at the beginning of the proof, we applied a rigid motion to $\hat{\gamma}$ to ensure that they start at the same point and in the same direction – so the conclusion is that the original curves differ by a rigid motion.)

- The proof of part (ii) is similar but more involved, and makes use of the existence and uniqueness theorem for systems of Ordinary Differential Equations.
- We will only mention that the relevant system of ODEs is the *Frenet-Serret equations* mentioned above. The fundamental theorem of ODEs says that there is a unique solutions to that system with given initial conditions, and this implies the existence of a unique curve with the prescribed κ and τ up to a proper rigid motion.

Exercises

- Tapp 1.69

EXERCISE 1.69. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a regular space curve (possibly with points where $\kappa = 0$ and hence where τ is undefined). *Prove or disprove*:

(1) If the trace of $\boldsymbol{\gamma}$ lies in a plane, then τ equals zero everywhere it is defined.

(2) If τ equals zero everywhere it is defined, then the trace of $\boldsymbol{\gamma}$ lies in a plane.

Math 435: Lecture 36

April 12, 2024

Reference: Tapp, pp. 62-71

Topics:

- Two famous theorems about simple closed curves.
  - Theorem 2.1:
    (1) Hopf's Umlaufsatz: the rotation index of any simple closed plane curve is ± 1.
    (2) The Jordan curve theorem: If γ is a simple closed plane curve with trace C, then $\mathbb{R}^2 - C$ has two path components U_{in} and U_{out}, where U_{in} is bounded and U_{out} is unbounded.
    This means (i) $\mathbb{R}^2 - C = U_{\text{in}} \cup U_{\text{out}}$ and $U_{\text{in}} \cap U_{\text{out}} = \emptyset$, (ii) each of these sets U is *path-connected*, meaning that any two points $\mathbf{p}, \mathbf{q} \in U$ are connected by a curve lying entirely in U, and (iii) any curve γ joining a point U_{in} and a point in U_{out} must intersect C.
- Rough sketch of proof of Umlaufsatz.
  - The proof relies on the notion of *winding number* of a continuous function $F\colon [a,b] \to \mathrm{S}^1$ with $F(a) = F(b)$, which is a generalization of the rotation index that we have already defined.
    Namely, just as we defined an angle function θ for a regular curve γ, having the property that $\vec{t}(t) = (\cos\theta(t), \sin\theta(t))$ for all t, one can more generally prove the existence of an angle function for *any* continuous function $F\colon I \to \mathrm{S}^1$, again with the property that $F(t) = (\cos\theta(t), \sin\theta(t))$ for all t.
    Then, given $F\colon [a,b] \to \mathrm{S}^1$ with $F(a) = F(b)$ with angle function θ, we define its winding number to be $\frac{1}{2\pi}(\theta(b) - \theta(a))$.
    Hence, we see that the rotation index of a simple closed curve γ is the winding number of its unit tangent vector $\vec{t}$.
  - Now suppose $\gamma\colon [a,b] \to \mathbb{R}^2$ is a simple closed curve, which we may suppose is unit-speed. Our goal is to show that the winding number of $\vec{t}$ is ± 1.
    Let $T \subset [a,b] \times [a,b]$ be the triangle $= T\{(t_1, t_2) \mid a \le t_1 \le t_2 \le b\}$.
    We define a continuous function $\psi\colon T \to \mathrm{S}^1$ as follows:

$$\psi(t_1, t_2) = \begin{cases} \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} & t_1 < t_2 \text{ and } (t_1, t_2) \neq (a,b) \\ \gamma'(t_1) & t_1 = t_2 \\ -\gamma'(t_a) & (t_1, t_2) = (a,b). \end{cases}$$

    We observe that the unit-tangent vector function for γ is given by $\vec{t}(t) = \psi(t,t)$.
  - Now define $\alpha_0, \alpha_1\colon [a,b] \to T$ by setting $\alpha_0(t) = (t,t)$ (so that $\vec{t} = \psi \circ \alpha_0$) and by letting α_1 be a parametrization of the union of the line segments from (a,a) to (a,b) and from (a,b) to (b,b).
  - The claim now follows from the following three facts:
    (1) There is a continuous deformation $\{\alpha_s\}_{s\in[0,1]}$ from α_0 to α_1 – namely $\alpha_s(t) = (1-s)\alpha_0(t) + s\alpha_1(t)$ – and hence a continuous deformation from $\vec{t} = \psi \circ \alpha_0$ to $\psi \circ \alpha_1$ (namely $\{\psi \circ \alpha_s\}_{s\in[0,1]}$).

(2) If two maps $F_0, F_1 \colon [a,b] \to \mathrm{S}^1$ with $F_0(a) = F_0(b)$ and $F_1(a) = F_1(b)$ are continuous deformations of one another, then they have the same winding number.
(3) $\psi \circ \alpha_1$ has winding number ± 1.

- Even rougher sketch of proof of Jordan curve theorem.
  - The Jordan curve theorem relies on the notion of a *tubular neighbourhood*: given a simple closed curve $\gamma \colon [a,b] \to \mathbb{R}^2$, we can define a function $F \colon [a,b] \times \mathbb{R} \to \mathbb{R}^2$ by $F(t,s) = \gamma(t) + s \cdot R_{90}\vec{t}(t)$.
  - If we choose ϵ small enough, then the restriction of F to $[a,b) \times (-\varepsilon, \varepsilon)$ is injective and its image is an open subset of $\mathbb{R}^2$ called a *tubular neighbourhood* of $C = \gamma([a,b])$. (This follows from the inverse function theorem.)
  - We can now see that there are *at most* two path components of $\mathbb{R}^2 - C$ as follows.
  - Fix two points $\mathbf{p}_1, \mathbf{p}_2$ in the tubular neighbourhood on opposite sides of γ, say $\mathbf{p}_1 = \gamma(a, \varepsilon/2)$ and $\mathbf{p}_2 = \gamma(a, -\varepsilon/2)$, and note that any point in the tubular neighbourhood can be connected via a path in $\mathbb{R}^2 - C$ either to $\mathbf{p}_1$ or $\mathbf{p}_2$.
  - Now consider any $\mathbf{p} \in \mathbb{R}^2 - C$ and consider a shortest path connecting $\mathbf{p}$ to γ. Before this path intersects γ, it must first intersect the tubular neighbourhood in some point $\mathbf{q}$. But since $\mathbf{q}$ can be connected to either $\mathbf{p}_1$ or $\mathbf{p}_2$ inside $\mathbb{R}^2 - C$, it follows that $\mathbf{q}$ can as well.
  - The proof that the two path-components are distinct follows from considering the function $W \colon \mathbb{R}^2 - C \to \mathbb{Z}$ taking $\mathbf{p}$ to the winding number of the function $f_{\mathbf{p}} \colon [a,b] \to \mathrm{S}^1$ given by $f_{\mathbf{p}}(t) = \frac{\mathbf{p} - \gamma(t)}{|\mathbf{p} - \gamma(t)|}$.
  - The claim then follows from the following facts:
    (1) If $\mathbf{p}_1$ and $\mathbf{p}_2$ are in the same path-component, then $W(\mathbf{p}_1) = W(\mathbf{p}_2)$ (since W is integer-valued, this follows by showing that W is continuous).
    (2) If $\mathbf{p}_1, \mathbf{p}_2$ are in the tubular neighbourhood and on opposite sides of C as above, then $|W(\mathbf{p}_1) - W(\mathbf{p}_2)| = 1$.
  - The claim that one of the components is bounded and the other isn't follows from the fact that if $\mathbf{p}$ is sufficiently far from γ, then $W(\mathbf{p}) = 0$.
- Umlaufsatz for piecewise-smooth curves
  - The Jordan curve theorem in fact holds for arbitrary *continuous* simple closed curves γ (i.e., continuous maps $\gamma \colon [a,b] \to \mathbb{R}^2$ that are injective on $[a,b)$ and with $\gamma(a) = \gamma(b)$), though the above proof doesn't work.
  - The Umlaufsatz doesn't make sense for continuous curves γ in general, since it refers to the tangent vector of γ.
  - However, there is a generalization of it for *piecewise-smooth simple closed curves*, i.e., continuous closed curves $\gamma \colon [a,b] \to \mathbb{R}^2$ such that there are finitely many points $a = t_0 < t_1 < \ldots < t_k = b$ such the restriction of γ to each $[t_i, t_{i+1}]$ is smooth. The points $\gamma(t_i)$ are called the *corners* of γ.
  - To state the theorem, define the *signed turning angle* α_i at each corner $\gamma(t_i)$, by consider the incoming and outgoing unit tangent vectors $\vec{t}_i^{\,-} = \vec{t}^{\,-}(t_i) = \lim_{t \nearrow t_i} \vec{t}(t)$ and $\vec{t}_i^{\,+} = \vec{t}^{\,+}(t_i) = \lim_{t \searrow t_i} \vec{t}(t)$, and defining $\alpha_i \in [-\pi, \pi]$ so that $|\alpha_i| = \angle(\vec{t}_i^{\,-}, \vec{t}_i^{\,+})$ and so that the sign of α_i is the sign of $\langle R_{90}\vec{t}_i^{\,-}, \vec{t}_i^{\,+} \rangle$ (i.e., it's positive if you need to turn counter-clockwise to get from $\vec{t}_i^{\,-}$ to $\vec{t}_i^{\,+}$).
  - Finally, say that γ is *positively oriented* if $R_{90}\gamma'(t)$ points toward the bounded component of $\mathbb{R}^2 - \gamma([a,b])$ whenever $t \neq t_i$ (and otherwise *negatively oriented*). We note

that a positively oriented *smooth* simple closed curve has rotation degree 1 (rather than -1).

- The theorem is then: Theorem 2.7: if γ is positively oriented, then

$$\int_a^b \kappa_{\mathrm{s}}(t) + \sum_{i=1}^n \alpha_i = 2\pi.$$

(Note that this integral is well-defined, since $\kappa_{\mathrm{s}}(t)$ is well-defined and continuous outside of the t_i.)

- Note that the "opposite" case from the case of a smooth γ is that in which γ is a polygon, i.e., each smooth segment is simply a line segment. Then the theorem just says that the sum of the signed turning angles of a polygon are equal to 2π.
- The idea of the proof is to reduce to the case of a smooth curve by "rounding the corners": we choose $\varepsilon > 0$ very small and replace γ by a smooth curve $\hat{\gamma}$ which is equal to γ outside of each $(t_i - \varepsilon, t_i + \varepsilon)$, and on these intervals, it is given by a small arc.
- Then the total signed curvature $\int_a^b \hat{\kappa}_{\mathrm{s}}$ of $\hat{\gamma}$ is 2π since $\hat{\gamma}$ is smooth. On the other hand, this differs from $\int_a^b \hat{\kappa}_{\mathrm{s}}$ by the sum of the $\int_{t_i-\varepsilon}^{t_i+\varepsilon} \hat{\kappa}_{\mathrm{s}}$, which is $\hat{\theta}(t_i + \varepsilon) - \hat{\theta}(t_i - \varepsilon)$, which converges to α_i as $\varepsilon \to 0$.
- Finally, we mention a reformulation in terms of the *interior angles* $\beta_i = \pi - \alpha_i$ (in the negatively oriented case, they would be $\pi + \alpha_i$).
- Then the theorem becomes

$$\int_a^b \kappa_{\mathrm{s}} = \left(\sum_{i=1}^n \beta_i\right) - (n-2)\pi$$

and in the particular case of a polygon, becomes the well-known formula

$$\sum_{i=1}^n \beta_i = (n-2)\pi.$$

Exercises

- Tapp 1.81

EXERCISE 1.81. Give examples of a regular plane curve $\boldsymbol{\gamma} : I \to \mathbb{R}^2$ and a rigid motion f of $\mathbb{R}^2$ such that $f \circ \boldsymbol{\gamma}$ is a reparametrization of $\boldsymbol{\gamma}$. Include an example in which the curve is closed and one in which it is not closed. Include examples in which the rigid motion is proper and in which it is improper.

Math 435: Lecture 37

April 15, 2024

Reference: Tapp, pp. 125-135, 167-169

Topics:

- Isometries between surfaces
  - We now begin our discussion of surfaces, i.e., 2-manifolds, in $\mathbb{R}^3$. Some of this generalizes to arbitrary manifolds, so we might as well discuss it in this generality, but you should keep in mind the case of surfaces throughout the discussion. (Note that Tapp uses the term *regular surface* for a 2-manifold in $\mathbb{R}^3$.)
  - Recall that each tangent space $\mathrm{T}_{\mathbf{p}}S$ of a manifold $M \subset \mathbb{R}^n$ has an inner product $\langle -, - \rangle \colon \mathrm{T}_{\mathbf{p}}M \times \mathrm{T}_{\mathbf{p}}M \to \mathbb{R}$, namely just the restriction of the standard inner product on $\mathbb{R}^n$. Sometimes we may denote it by $\langle -, - \rangle_{\mathbf{p}}$ to emphasize that it is an inner product on the space $\mathrm{T}_{\mathbf{p}}M$.
    Hence, we also have the norm map $|-|_p = |-| \colon \mathrm{T}_pM \to \mathbb{R}$.
    As always, each of these determines the other via $\langle \vec{u}\,, \vec{w}\,\rangle_{\mathbf{p}} = \frac{1}{2}(|\vec{u}\, + vw|^2_{\mathbf{p}} - |\vec{u}\,|^2_{\mathbf{p}} - |\vec{w}\,|^2_{\mathbf{p}})$.
  - A diffeomorphism $f \colon M_1 \to M_2$ between manifolds is an *isometry* if its derivative $\mathrm{d}f_{\mathbf{p}} \colon \mathrm{T}_{\mathbf{p}}M_1 \to \mathrm{T}_{f(\mathbf{p})}M_2$ at each point (remember, we are no longer using Spivak's notation f_*!) preserves the norm (or equivalently, the inner product): $\|\mathrm{d}f_{\mathbf{p}}\vec{u}\,\|_{f(\mathbf{p})} = \|\vec{u}\,\|_{\mathbf{p}}$ for all $\vec{u} \in \mathrm{T}_{\mathbf{p}}M_1$.
  - Two manifolds are *isometric* if there is an isometry between them.
  - The most obvious examples are: if f is a rigid motion of $\mathbb{R}^n$, then the restriction to any manifold $f \colon M \to f(M)$ is an isometry.
  - In this example if $M = f(M)$, then f induces an isometry from M to itself; this is called an *isometry of* M. An example is a rotation of a sphere.
  - The more interesting examples of isometries do not come from rigid motions. An example is the map $f \colon (-\pi, \pi) \times \mathbb{R} \to C \subset \mathbb{R}^3$ given by $f(u, v) = (\cos u, \sin u, v)$ from an infinite strip to a slit cylinder $C = f(V) = \{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } x \neq -1\}$. We have $\mathrm{d}f_{(u,v)}(\vec{e}_1) = \frac{\partial f}{\partial u}(u, v) = (-\sin u, \cos u, 0)$ and $\mathrm{d}f_{(u,v)}(\vec{e}_2) = \frac{\partial f}{\partial v}(u, v) = (0, 0, 1)$, hence $\mathrm{d}f_{(u,v)}$ takes an orthonormal basis to an orthonormal basis, and hence preserves the inner product.
  - (Aside: this is a general fact: if $T \colon V \to W$ is a linear map between inner product spaces and $\vec{u}_1, \ldots, \vec{u}_n$ is an orthonormal basis of V and $\{f(\vec{u}_1), \ldots f(\vec{u}_n)\} \subset W$ is also an orthonormal set, then f preserves the inner product, since if $\vec{w} = \sum_i a_i \vec{u}_i$, then $|vw|^2 = \sum_i a_i^2$, and $|f(\vec{w}\,)|^2 = |\sum_i a_i f(\vec{u}_i)|^2 = \sum_i a_i^2$.)
  - A slightly more natural example is the "wrapping" map $f \colon \mathbb{R}^2 \to D$ given by the same formula, where now D is the (non-slit) cylinder. This map still preserves the inner product, but is now only a *local* diffeomorphism (i.e., it restrict to a diffeomorphism is a neighbourhood of each point). Such a map is called a *local isometry.*
  - As you will show in an exercise, a diffeomorphism $f \colon M_1 \to M_2$ is an isometry if and only if it preserves the lengths of curves, i.e., if for all $\gamma \colon [a, b] \to M_1$, we have $\mathrm{length}(\gamma) = \mathrm{length}(f \circ \gamma)$.
  - Similarly, for two regular (say, unit-speed) curves γ_1, γ_2 meeting at a point $\mathbf{p} = \gamma_1(t_0) = \gamma_2(t_1)$, their angle at that point $\cos^{-1}(\langle \gamma_1'(t_0), \gamma_2'(t_1) \rangle)$ is also preserved by f.

  - We call a property of a manifold "intrinsic" if it is preserved by arbitrary isometries. Intuitively, this is a property that only depends on measurements of lengths and angles on the surface, and on how the surface is embedded in $\mathbb{R}^3$.
- Surface patches and parametrized surfaces
  - Given a surface S, a *surface patch* is another word for a parametrization $\sigma\colon U \to V \subset S$, i.e., a diffeomorphism where $U \subset \mathbb{R}^2$ is open and $V \subset S$ is an open subset of S (i.e., an intersection $V' \cap S$ for some open subset $V' \subset \mathbb{R}^3$).
  - We typically denote surface patches by σ, and if $u, v\colon U \to \mathbb{R}$ are the coordinates on U, we write $\sigma_u\colon U \to \mathbb{R}^3$ and $\sigma_v\colon U \to \mathbb{R}^3$ for the partial derivative of σ. Note that $\sigma_u(\mathbf{q}) = \mathrm{d}\sigma_{\mathbf{q}}(\vec{e}_1) \in \mathrm{T}_{\sigma(\mathbf{q})}S$ and similarly $\sigma_v(\mathbf{q}) \in \mathrm{T}_{\sigma(\mathbf{q})}$ for all $\mathbf{q} \in U$.
  - The curves $\sigma(u, v_0)$ for fixed v_0 and $\sigma(u_0, v)$ for fixed u_0 on S are called *parameter curves*. The vectors σ_u and σ_v are precisely the tangent vectors to the parameter curve.
  - Recall that (by the inverse function theorem) given any smooth map $\sigma\colon U \to \mathbb{R}^3$ with $U \subset \mathbb{R}^2$ open and $\mathbf{q} \in U$, if $\mathrm{d}\sigma_{\mathbf{q}}$ is injective (i.e., rank 2), then $\mathbf{q}$ has some neighbourhood V such that $\sigma(V)$ is a surface and $\sigma|_V\colon V \to \sigma(V)$ is parametrization.
  - Tapp calls such a map σ a *parametrized surface*.
  - Note that the condition for σ to be a parametrized surface is precisely that σ_u and σ_v are independent, or equivalently that $\sigma_u \times \sigma_v \neq 0$.

Exercises

- Tapp 3.18

EXERCISE 3.18 (**Generalized Cylinders**).

(1) Let $\gamma : [a, b] \to \mathbb{R}^3$ be a simple closed space curve whose trace lies in a plane $P \subset \mathbb{R}^3$. Let $n \in \mathbb{R}^3$ be a normal vector to P. Prove that

$$C = \{\gamma(t) + sn \mid t \in [a, b], s \in \mathbb{R}\}$$

is a regular surface that is diffeomorphic to the standard cylinder from Example 3.23. The set C is called a **generalized cylinder**. Notice that C in unaffected by reparametrizing γ or scaling n, so we can assume without loss of generality that γ is of unit speed and that n is of unit length. *HINT: After applying a rigid motion, you can assume that P is the xy-plane. Use Exercise 3.11 (on page 124) to establish the smoothness of the inverse of the natural surface patch.*

- Tapp 3.83

EXERCISE 3.83. Show that every generalized cylinder (Exercise 3.18(1) on page 136) is isometric to a standard cylinder of the form

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = R^2\}$$

Math 435: Lecture 38

April 17, 2024

Reference: Tapp, pp. 131, 165-166, 182-185

Topics:

- Surface patches for S^2
  - As an example, a surface patch for the 2-sphere S^2 is given by $\sigma\colon (0,2\pi)\times(0,\pi)\to \mathrm{S}^2$ defined by
    $$\sigma(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$
  - We have $\sigma_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0)$ and $\sigma_\varphi = (\cos\theta\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$. Thus $|\sigma_\theta\times\sigma_\varphi| = \sin\varphi \neq 0$ so $\mathrm{d}\sigma_{\mathbf{q}}$ is indeed injective for all $\mathbf{q}$, so σ is a parametrized surface.
  - σ is clearly injective, hence bijective onto its image, which is $C = \mathrm{S}^2 - \{(x,0,z) \mid x\geq 0\}$. To see that is a surface patch (i.e., a diffeomorphism onto its image), one needs to check that its inverse is smooth; this amounts to solving for (φ,θ) in terms of $(x,y,z)\in C$. We have $\varphi = \arccos z$ and $(x,y) = F(x/\sin\varphi, y/\sin\varphi)$ where $F\colon \mathrm{S}^1 - \{(0,1)\}$ is inverse of the diffeomorphism $(\cos t,\sin t)\colon (0,2\pi)\to \mathrm{S}^1$.
  - The parameter curves $\sigma(\theta,\varphi_0)$ and $\sigma(\theta_0,\varphi)$ are the parallels and meridians, respectively.
  - To obtain an atlas for S^2, we need to use a second surface patch which instead excludes a different half circle, obtained for example by composing the above surface patch with a suitable rotation in $\mathbb{R}^3$
- Surfaces of revolution
  - Here is a nice source of examples of surfaces: begin with a curve $\gamma(t) = (x(t), 0, z(t))$, $t\in(a,b)$ in the xz-plane, and assume $x>0$ and $z'>0$.
  - Let C be the trace of γ and let S the result of revolving C around the z-axis, i.e., the image of $\sigma(\theta,t) = R_\theta(\gamma(t)) = (x(t)\cos\theta, x(t)\sin\theta, z(t)), \theta\in\mathbb{R}, t\in(a,b)$. Here R_θ is the rotation by angle θ around the z-axis, which is represented by the matrix
    $$\begin{bmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}.$$
  - Note that by assumption, the function $z=\gamma_3$ is a diffeomorphism, so we have a smooth inverse $\gamma_3^{-1}\colon (\gamma_3(a),\gamma_3(b))\to(a,b)$. We see that S is exactly the regular level set $G^{-1}(0)$ for the smooth function $G(x,y,z) = x^2+y^2-\gamma_1(\gamma_3^{-1}(z))^2$. This shows that S is indeed a surface.
  - We have that σ is a parametrized surface since $\sigma_\theta = (-x\sin\theta, x\cos\theta, 0)$ and $\sigma_t = (x'\cos\theta, x'\sin\theta, z')$, which are clearly independent since $x, z'>0$. Hence the restriction of σ to any sufficiently small open set (in fact, any set of the form $(\theta_0,\theta_0+2\pi)\times(a,b)$) is a surface patch.
- Area is intrinsic
  - We have seen that lengths of curves and angles are intrinsic. Areas are also intrinsic, i.e., given an isometry $f\colon S_1\to S_2$, say, with S_1 compact, and a bounded open set $U\subset S_1$, we have $\mathrm{Area}(U) = \mathrm{Area}(f(U))$.

  - To see this, recall that we have defined the area of U to be $\int_U \mathrm{d}V$, where $\mathrm{d}V$ is the volume form, which for a surface is given by $\mathrm{d}V(\vec{u}, \vec{v}) = \langle \vec{n}(x), vu \times \vec{v} \rangle$, where $\vec{n}(x)$ is the unit normal vector to the surface.
  - (This depends on the chosen orientation but – as Spivak shows in Exercises 5-21 and 5-27, but which we didn't discuss (though it was raised in class as a question) – the absolute value $|\mathrm{d}V|(\vec{u}, \vec{v}) = |\vec{u} \times \vec{v}|$ doesn't depend on the orientation, and it turns out one cans still define an integral of such an "absolute k-form" on a k-manifold, so that the volume still makes sense, independent of orientation or orientability.)
  - In particular, for a surface patch $\sigma\colon W \xrightarrow{\sim} U$, we have $\mathrm{Area}(U) = \int_W |\sigma_u \times \sigma_v| \,\mathrm{d}u \,\mathrm{d}v$.
  - To see that this is preserved by isometries, we may simply note that, if $f\colon S_1 \to S_2$ is an isometry, then $f^*|\mathrm{d}V_{S_2}| = |\mathrm{d}V_{S_1}|$ (where these are the respective volume forms of S_1 and S_2). This is because the volume form was defined to be the unique 2-form taking the value 1 on a(n oriented) orthonormal basis. But since f is an isometry, $\mathrm{d}f_{\mathbf{p}}\colon \mathrm{T}_{\mathbf{p}}S_1 \to \mathrm{T}_{f(\mathbf{p})}S_2$ is an isomorphism of inner product spaces for each $\mathbf{p} \in S_1$, and hence takes an orthonormal basis to an orthonormal basis.
  - Somewhat more concretely, if $\sigma\colon W \xrightarrow{\sim} U$ is a surface patch, then $|\sigma_u \times \sigma_v| = |\sigma_u| \cdot |\sigma_v| \cdot \sin \angle(\sigma_u, \sigma_v)$ is preserved by f, since f preserves lengths of and angles between tangent vectors.
- The first fundamental form of a surface
  - The *first fundamental form* of a surface $S \subset \mathbb{R}^3$ is the function assigning to each $\mathbf{p} \in S$, the squared-norm function $|-|^2_{\mathbf{p}}$.
    (Sometimes, it instead refers to the function assigning to each $\mathbf{p}$ the inner product $\langle -, - \rangle_{\mathbf{p}}$).
  - It is something like a 1-form, since it is assigns a function $\mathrm{T}_{\mathbf{p}} \to \mathbb{R}$ to each tangent space, except that now these are quadratic and not linear functions.
  - (Similarly, $\mathbf{p} \mapsto \langle -, - \rangle_{\mathbf{p}}$ is something like a 2-form, except that $\langle -, - \rangle_{\mathbf{p}}$ is symmetric rather than alternating.)
  - It is interesting to see what the first fundamental form looks like in local coordinates.
  - Thus, given a parametrization $\sigma\colon U \to V \subset S$, we may *pull the first fundamental form* back along σ, i.e., consider the function $\mathcal{F}_1$ assigning to each $\mathbf{p} \in U$ the function $\mathrm{T}_{\mathbf{p}}U = \mathbb{R}^2 \to \mathbb{R}$ given by $\vec{v} \mapsto |\mathrm{d}\sigma_{\mathbf{q}}(\vec{v})|^2_{\sigma(\mathbf{q})}$.
  - Let us define functions $E = |\sigma_u|^2$, $F = \langle \sigma_u, \sigma_v \rangle$, $G = |\sigma_v|^2$.
  - We then have, for any $\mathbf{q} \in U$ and $\vec{v} = (v_1, v_2) = (\mathrm{d}u(\vec{v}), \mathrm{d}v(\vec{v})) \in \mathbb{R}^2$, that $\mathrm{d}\sigma_{\mathbf{p}}(\vec{v}) = v_1\sigma_u(\mathbf{p}) + v_2\sigma_v(\mathbf{p})$ and hence

$$\begin{aligned} \mathcal{F}_1(\mathbf{p})(\vec{v}) &= \|v_1\sigma_u(\mathbf{p}) + v_2\sigma_v(\mathbf{p})\|^2 \\ &= v_1^2\|\sigma_u(\mathbf{p})\|^2 + 2v_1v_2\langle \sigma_u(\mathbf{p}), \sigma_v(\mathbf{p}) \rangle + v_2^2\|\sigma_v(\mathbf{p})\|^2 \\ &= (E(\mathbf{p})\,\mathrm{d}u^2 + 2F(\mathbf{p})\,\mathrm{d}u\,\mathrm{d}v + G(\mathbf{p})\,\mathrm{d}v^2)(\vec{v}). \end{aligned}$$

  - We thus write

$$\mathcal{F}_1 = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2.$$

  - Put another way, the function E, F, G are such that at each $\mathbf{p} \in U$, the bilinear form $\vec{u}, \vec{v} \mapsto \langle \mathrm{d}\sigma_{\mathbf{p}}(\vec{u}), \mathrm{d}\sigma_{\mathbf{p}}(\vec{v}) \rangle$ is given by

$$\vec{u}^{\top} \cdot \begin{bmatrix} E(\mathbf{p}) & F(\mathbf{p}) \\ F(\mathbf{p}) & G(\mathbf{p}) \end{bmatrix} \cdot \vec{v}.$$

– We can compute lengths and areas in local coordinates using the first fundamental form.

– If $\gamma\colon [0,l] \to U$ is a regular curve, then writing $\gamma(t) = (u(t), v(t))$, we have

$$\begin{aligned}\operatorname{length}(\sigma\circ\gamma) &= \int_0^l \sqrt{\mathcal{F}_1} = \int_0^l \sqrt{E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2}\\ &= \int_0^l \sqrt{E\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 + 2F\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + G\Big(\frac{\mathrm{d}v}{\mathrm{d}t}\Big)^2}\,\mathrm{d}t\end{aligned}$$

– Similarly, the area of $\sigma(R) \subset S$ for a region $R \subset U$ contained in compact subset of U is given by

$$\int_R |\sigma_u \times \sigma_v| = \int_R \sqrt{|\sigma_u|^2|\sigma_v|^2 - \langle\sigma_u,\sigma_v\rangle^2} = \int_R \sqrt{EG - F^2}.$$

– Example, for the spherical coordinate chart $\sigma\colon (0,2\pi)\times(0,\pi) \to \mathrm{S}^2$ given by $\sigma(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\theta)$, we had

$$\sigma_\theta = (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0) \quad \text{and} \quad \sigma_\varphi = (\cos\theta\cos\theta, \cos\varphi\sin\theta, -\sin\varphi)$$

– We thus have

$$E = |\sigma_\theta|^2 = \sin^2\varphi \qquad F = \langle\sigma_\theta,\sigma_\varphi\rangle = 0 \qquad G = |\sigma_\varphi|^2 = 1.$$

– Hence the first fundamental form is

$$E\,\mathrm{d}\theta^2 + 2F\,\mathrm{d}\theta\,\mathrm{d}\varphi + G\,\mathrm{d}\varphi^2 = \sin^2\varphi\,\mathrm{d}\theta^2 + \mathrm{d}r\varphi^2.$$

Exercise

- Compute the first fundamental form of the surface patch $\sigma(\theta,t) = (x(t)\cos\theta, x(t)\sin\theta, z(t))$ of the surface of revolution of $\gamma(t) = (x(t), 0, z(t))$.

Math 435: Lecture 39

April 19, 2024

Reference: Tapp, pp. Chapter 4

Topics:

- Curvature on surfaces
  - To study curvature on surfaces, we will have to familiarize ourselves with four related objects, and understand the relationship between them: the *Gauss map*, the *Weingarten map*, and the *second fundamental form*.
  - These will give rise to three notion of curvature: the *normal curvature* (of a curve on a surface), and the *mean curvature* and *Gaussian curvature* (of the surface itself). Of these, the Gaussian curvature is the most important.
  - To discuss curvature on surface S, we will first want to assume S is *oriented*. It will then turn out that the most important notion (the Gaussian curvature) is independent of the orientation). Also, note that *locally* (i.e., in some small neighbourhood around any point), any surface can be oriented.
  - Recall that an orientation for S amounts to specify a normal vector field $\vec{N} : S \to \mathbb{R}^3$ to S, i.e., a smooth function with $\vec{N}(\mathbf{p}) \in \mathrm{T}_{\mathbf{p}}(S)^{\perp}$ and $|\vec{N}(\mathbf{p})| = 1$ for all $\mathbf{p} \in S$.
  - Given a unit-speed curve γ on an oriented surface S, its *normal curvature* is the quantity $\kappa_{\mathrm{n}} = \langle \gamma''(t), \vec{N}(\gamma(t))\rangle$. Hence, it is positive if the γ is bending away from the normal vector, and negative otherwise.
  - Since $\gamma' \perp \vec{N}$, we have

    $$\kappa_{\mathrm{n}}(t) = \langle \gamma''(t), (\vec{N} \circ \gamma)(t)\rangle = -\langle \gamma'(t), (\vec{N} \circ \gamma)'(t)\rangle = \langle \gamma'(t), -\mathrm{d}\vec{N}_{\gamma(t)}(\gamma(t))\rangle.$$
  - Thus, we see that the normal curvature of γ at $\mathbf{p} = \gamma(t)$ actually only depends on $\vec{v} = \gamma'(t)$ – if we had a second curve α with $\alpha(t) = \mathbf{p}$ and $\alpha'(t) = \vec{v}$, it would have the same Gaussian curvature at $\mathbf{p}$, namely $\langle \vec{v}, -\mathrm{d}\vec{N}_{\mathbf{p}}\rangle$.
  - Thus, for $\mathbf{p} \in S$ and $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$, we define the *normal curvature at S in the direction (of a normal vector) $\vec{v}$* to be the quantity $\mathrm{II}_{\mathbf{p}}(\vec{v}) := \langle -\mathrm{d}\vec{N}_{\mathbf{p}}, \vec{v}\rangle$. (We will explain the notation once we introduce the second fundamental form.) Note that this is the rate at which the normal vector is bending as we move in the direction $\vec{v}$ from $\mathbf{p}$.
  - At each point $\mathbf{p}$, we can consider the *minimal* and *maximal* normal curvature at $\mathbf{p}$. These are called the *principal curvatures*:

    $$k_1 = \min_{\substack{\vec{v} \in \mathrm{T}_{\mathbf{p}}S \\ |\vec{v}|=1}} \mathrm{II}_{\mathbf{p}}(\vec{v}) \qquad k_2 = \max_{\substack{\vec{v} \in \mathrm{T}_{\mathbf{p}}S \\ |\vec{v}|=1}} \mathrm{II}_{\mathbf{p}}(\vec{v}).$$

    Note that these may be both positive, both negative, or have opposite signs.
  - The *Gaussian curvature* at $\mathbf{p}$ is the *product* of the principal curvatures: $K(p) = k_1 k_2$.
  - The *mean curvature* at $\mathbf{p}$ is the *average* of the principal curvatures: $H(p) = \frac{1}{2}(k_1 + k_2)$.
  - If we switch the orientation of S, then we change the sign of the normal curvature of all curves, hence of the principal curvatures, and hence of the mean curvature. However, the Gaussian curvature remains the same, hence is independent of orientations.
  - Example: the Gaussian curvature on a sphere of radius R is constantly $1/R^2$ since all normal curvatures, hence both principal curvatures, at any point are equal to $1/R$. It also constant mean curvature $\pm 1/R$ (depending on the orientation).

- In general, a point on a surface with $k_1 = k_2$ is called an *umbilical point.*
- The point $(a, 0, 0)$ on an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is not umbilical, but it clearly still has *positive* Gaussian curvature $K > 0$.
- The point $(0, 0, 0)$ on the hyperbolic paraboloid $z = x^2 - y^2$ has negative Gaussian curvature $K < 0$. By symmetry considerations, one can see that the mean curvature at this point must be 0.
- A plane obviously has constant zero Gaussian curvature (and mean curvature).
- A *cylinder* also has constant zero Gaussian curvature, since each point has a curve with normal curvature 0 and (exercise:) all the normal curvature of all other curves have the same sign.

  We also see that the *mean curvature* of a cylinder is *not* zero.
- Hence, the mean curvature cannot be intrinsic, since we know that the plane and the cylinder are (locally) isometric.
- However, the Gaussian curvature still has a change of being intrinsic – and in fact, it is!

  This is Gauss' *Theorema Egregium* (amazing theorem), so called because he was so surprised and delighted by it – intuitively, one would not expect any notion of curvature to be intrinsic.

Exercise:

- Show that all normal curvatures on a cylinder at a given point have the same sign.
- Find a surface which has points of positive, zero, and negative Gaussian curvature.

Math 435: Lecture 40

April 22, 2024

Reference: Tapp, pp. Chapter 4

Topics:

- The Gauss and Weingarten maps, and the second fundamental form
  - We will now explain some important reformulations of the principal and Gaussian curvatures.
  - First, the *Gauss map* of an oriented surface S is another name for the unit normal vector map $\vec{N}: S \to \mathbb{R}^3$, except that we regard it as a smooth map of surfaces $\vec{N}: S \to \mathrm{S}^2$.
  - Hence, for each $\mathbf{p} \in S$, we can consider its derivative $\mathrm{d}\vec{N}_{\mathbf{p}} \colon \mathrm{T}_{\mathbf{p}}S \to \mathrm{T}_{\vec{N}(\mathbf{p})}\mathrm{S}^2$. But notice that the planes $\mathrm{T}_{\mathbf{p}}S$ and $\mathrm{T}_{\vec{N}(\mathbf{p})}\mathrm{S}^2$ are equal (recall that they are both considered as *linear subspaces* of $\mathbb{R}^3$, i.e., planes through the origin), since they both have normal vector $\vec{N}(\mathbf{p})$.
  - Hence we can consider $\mathrm{d}\vec{N}_{\mathbf{p}}$ as a map $\mathrm{T}_{\mathbf{p}}S \to \mathrm{T}_{\mathbf{p}}S$.
  - The *Weingarten map* (also called the *shape operator*) of S at $\mathbf{p}$ is the linear map $\mathcal{W}_{\mathbf{p}} = -\mathrm{d}\vec{N}_{\mathbf{p}} \colon \mathrm{T}_{\mathbf{p}}S \to \mathrm{T}_{\mathbf{p}}S$.
  - Note that by definition, the normal curvature at $\mathbf{p} \in S$ in the direction $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$ (with $\vec{v}$ a unit vector) is given by

(3) $$\mathrm{II}_{\mathbf{p}}(\vec{v}) = \langle \mathcal{W}_{\mathbf{p}}\vec{v}, \vec{v} \rangle.$$

  - More generally, we define the *second fundamental form* of $\mathcal{S}$ at $\mathbf{p}$ to be the function $\mathrm{II}_{\mathbf{p}}(\vec{v}) \colon \mathrm{T}_{\mathbf{p}}S \to \mathbb{R}$ given by (3) (i.e., we extend it to *all* tangent vectors, not only unit length ones). Like the first fundamental form, it assigns a *quadratic* function to each tangent space (in the sense that $\mathrm{II}_{\mathbf{p}}(a\vec{v}) = a^2\mathrm{II}_{\mathbf{p}}(\vec{v})$).
  - The fundamental property of $\mathcal{W}_{\mathbf{p}}$ is that it is *self-adjoint.* A linear operator $T \colon V \to V$ on an inner product space V is *self-adjoint* if $\langle T\vec{u}, \vec{v} \rangle = \langle \vec{u}, T\vec{v} \rangle$ for all $\vec{u}, \vec{v} \in V$.
  - On $\mathbb{R}^n$ with the standard inner product $\langle \vec{u}, \vec{v} \rangle = \vec{u}^{\top} \cdot \vec{v}$, this just means that the matrix representing T is *symmetric*: $T^{\top} = T$.
  - To see that $\mathcal{W}_{\mathbf{p}}$ is self-adjoint, note that it suffices by the bilinearity of $\langle \mathcal{W}_{\mathbf{p}} -, - \rangle$ to check $\langle \mathcal{W}_{\mathbf{p}}\vec{u}, \vec{v} \rangle = \langle \vec{u}, \mathcal{W}_{\mathbf{p}}\vec{v} \rangle$ with $\vec{u}, \vec{v}$ both vectors from any fixed *basis* $\vec{b}_1, \vec{b}_2$ of $\mathrm{T}_{\mathbf{p}}S$ (and moreover, it suffices to consider the case $\vec{u} = \vec{b}_1$ and $\vec{v} = \vec{b}_2$ since the equation obviously holds when $\vec{u} = \vec{v}$).

    Choosing a parametrization $\sigma \colon U \to V \subset S$ with $\sigma(\mathbf{q}) = \mathbf{p}$, we will prove it for the basis $\{\sigma_u(\mathbf{q}), \sigma_v(\mathbf{q})\}$.

    Now we evaluate
    $$0 = \frac{\partial}{\partial u}\langle \vec{N} \circ \sigma, \sigma_v \rangle = \langle \mathrm{d}\vec{N}(\sigma_u), \sigma_v \rangle + \langle N \circ \sigma, \sigma_{uv} \rangle.$$
    where the first equation is because $\vec{N}$ is perpendicular to S whereas σ_v is tangent to it.

    Switching the roles of u and v, we obtain $0 = \langle \mathrm{d}\vec{N}(\sigma_v), \sigma_u \rangle + \langle N \circ \sigma, \sigma_{uv} \rangle$, and hence we conclude $\langle \mathrm{d}\vec{N}(\sigma_u), \sigma_v \rangle = \langle \mathrm{d}\vec{N}(\sigma_v), \sigma_u \rangle$ as desired.

- Why is it good to know that $\mathcal{W}$ is self-adjoint? One reason is the fundamental property: for *any self-adjoint operator T, there is a basis consisting of eigenvectors for T.*
- To prove this, it suffices to prove it in the case $V = \mathbb{R}^n$ since every n-dimensional inner product space is isomorphic to $\mathbb{R}^n$.
- We prove it in the case of interest $\dim V = 2$. Recall that the eigenvalues of a matrix A are the roots of its *characteristic polynomial* $p(\lambda) = \det(\lambda I - A)$.
- Here, $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, and hence $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - b^2)$. This has discriminant $(a+d)^2 - 4(ad-b^2) = (a-d)^2 + b^2$, which is non-negative, hence both roots are *real*.
- Now, if both roots are equal (i.e., $a = d$ and $b = 0$), then A is just a diagonal matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, hence *every* vector is an eigenvector, and we can obviously choose an orthonormal basis.
- Otherwise, there are two distinct eigenvalues λ, μ, and we use the following general fact: *if $\vec{u}$ and $\vec{v}$ are eigenvectors for T with distinct eigenvalues λ and μ, then $\vec{u} \perp \vec{v}$.* Indeed,
$$\lambda\langle \vec{u}, \vec{v}\rangle = \langle T\vec{u}, \vec{v}\rangle = \langle \vec{u}, T\vec{v}\rangle = \mu\langle \vec{u}, \vec{v}\rangle$$
and hence $\langle \vec{u}, \vec{v}\rangle = 0$ if $\lambda \neq \mu$.
- Next, the *quadratic form* associated to a self-adjoint operator T is the function $Q(\vec{v}) = \langle T\vec{v}, \vec{v}\rangle$. It determines T since $\langle T\vec{u}, \vec{v}\rangle = \frac{1}{2}\big(Q(\vec{u}+\vec{v}) - Q(\vec{u}) - Q(\vec{v})\big)$ (and T is determined by the function $\vec{u}, \vec{v} \mapsto \langle T\vec{u}, \vec{v}\rangle$ since if $\vec{b}_1, \ldots, \vec{b}_n$ is an orthonormal basis, then $T\vec{u} = \sum_i \langle T\vec{u}, \vec{b}_i\rangle \vec{b}_i$).
- Thus, the quadratic form associated to $\mathcal{W}_{\mathbf{p}}$ is $\mathrm{II}_{\mathbf{p}}$.
- Now, in general, if $\vec{b}_1, \ldots, \vec{b}_n$ is an orthonormal basis of eigenvectors for T with eigenvalues $\lambda_1, \ldots, \lambda_n$, we have for $\vec{v} = \sum_{i=1}^n v_i \vec{b}_i$ that
$$Q(\vec{v}) = \langle \sum_{i=1}^n v_i \vec{b}_i, T\sum_{i=1}^n v_i \vec{b}_i\rangle = \sum_{i=1}^n \lambda_i v_i^2.$$
- Letting $\lambda_{\min}$ and $\lambda_{\max}$ be the minimal and maximal eigenvalues, it follows that *$\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and maximal values attained by $Q(\vec{v})$ on unit vectors $\vec{v}$.* (Indeed, $\sum_i \lambda_i v_i^2 \le \sum_i \lambda_{\max} v_i^2 = \lambda_{\max}$, and similarly $Q(\vec{v}) \ge \lambda_{\min}$.)
- We conclude that *the principal curvatures of S at p are the eigenvalues of $\mathcal{W}_{\mathbf{p}}$.*
- Moreover, the *principal directions* $\vec{v}_1, \vec{v}_2$, i.e., the unit vectors for which $\mathrm{II}_{\mathbf{p}}(\vec{v}_i) = \lambda_i$ are *orthogonal.*
- Some care must be taken about this at umbilical points (i.e., when $\lambda_1 = \lambda_2$). In this case, *every* direction is a principal direction (i.e., every vector is an eigenvector, and $\mathcal{W}_{\mathbf{p}}$ is just the operator $\mathcal{W}_{\mathbf{p}}(\vec{v}) = \lambda\vec{v}$) – but we can of course still *choose* two principal directions which are orthogonal.
- Finally, recall that for a linear operator $T\colon V \to V$ on a finite-dimensional vector space V, its *trace* $\mathrm{tr}(T)$ and *determinant* $\det(T)$ are defined to be the trace and determinant of the matrix A representing T with respect to any basis $\vec{b}_1, \ldots, \vec{b}_n$ of V.
- If we choose a different basis $\vec{b}'_1, \ldots, \vec{b}'_n$ and define the change of basis matrix P by $\vec{b}'_i = \sum_j P_{ij}\vec{b}_j$, the matrix of T with respect to the second basis is $A' = PAP^{-1}$. By the multiplicativity of det and the property $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ of the trace, it

follows that $\det(A) = \det(A')$ and $\mathrm{tr}(A) = \mathrm{tr}(A')$ and hence that $\det(T)$ and $\mathrm{tr}(T)$ are well-defined.

- Now, if $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$, then the matrix of T is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ and hence $\det(T) = \lambda_1 \cdots \lambda_n$ and $\mathrm{tr}(T) = \lambda_1 + \cdots + \lambda_n$.
- We conclude that the *Gaussian and mean curvature are the determinant and half the trace of the Weingarten map* $K(\mathbf{p}) = \det(\mathcal{W}_{\mathbf{p}})$ and $H(\mathbf{p}) = \frac{1}{2}\mathrm{tr}(\mathcal{W}_{\mathbf{p}})$.

Math 435: Lecture 41

April 24, 2024

Reference: Tapp, pp. Chapters 4 and 5

Topics:

- Second fundamental form in local coordinates
  - As with the first fundamental form, we can also represent the second fundamental form in local coordinates.
  - That is, given a surface patch $\sigma \colon U \xrightarrow{\sim} V \subset S$, we can consider the function $(\mathcal{F}_2)_{\mathbf{q}} \colon \mathbb{R}^2 = \mathrm{T}_{\mathbf{q}}U \to \mathbb{R}$ given by $(\mathcal{F}_2)_{\mathbf{q}}(\vec{v}\,) = \mathrm{II}_{\mathbf{p}}(\mathrm{d}\sigma_{\mathbf{q}}(\vec{v}\,))$.
  - As before, we can write this in the form

$$\mathcal{F}_2 = e\,\mathrm{d}u^2 + 2f\,\mathrm{d}u\,\mathrm{d}v + g\,\mathrm{d}v^2$$

    where now $e = \mathrm{II}(\sigma_u) = \langle \sigma_{uu}, \vec{N}\,\rangle$, $f = \mathcal{W}(\sigma_u, \sigma_v) = \langle \sigma_{uv}, \vec{N}\,\rangle$, and $g = \mathrm{II}(\sigma_v) = \langle \sigma_{vv}, \vec{N}\,\rangle$.
  - We then obtain such formulas as

$$K = \frac{eg - f^2}{EG - F^2} \qquad H = \frac{eG - 2fF + gE}{EG - F^2} \qquad \{k_1, k_2\} = \{H - \sqrt{H^2 - K}, H + \sqrt{H^2 - K}\}.$$

- Total Gaussian curvature
  - Given a plane curve $\gamma \colon [a,b] \to \mathbb{R}$, its *total signed curvature* $\int_a^b \kappa_{\mathrm{s}}\,\mathrm{d}t$ can be interpreted as the *signed length* of the curve $\vec{t} \colon [a,b] \to \mathrm{S}^1$, where by "signed" length we mean $\int_{[a,b]} \vec{t}^{\,*}\,\mathrm{d}V$, where $\mathrm{d}V$ is the volume form on the circle (with its standard orientation).
  - Indeed, the oriented unit tangent vector at $\vec{v} \in \mathrm{S}^1$ is precisely $R_{90}\vec{v}$, and hence

$$\vec{t}^{\,*}\,\mathrm{d}V(\vec{e}_1) = \mathrm{d}V(\vec{t}^{\,\prime}) = \langle \vec{t}^{\,\prime}, R_{90}\vec{t}\,\rangle = \kappa_{\mathrm{s}} = \kappa_{\mathrm{s}}\,\mathrm{d}t(\vec{e}_1).$$

  - The *total signed curvature* $\int_S K\,\mathrm{d}V_S$ of a surface can be given a similar interpretation, namely as the *signed area* $\int_S \vec{N}^{\,*}\,\mathrm{d}V_{\mathrm{S}^2}$ of the Gauss map $\vec{N} : S \to \mathrm{S}^2$.
  - (In fact, this was Gauss' original definition of K: as the "infinitesimal signed area of the Gauss map".)
  - In other words, the claim is that $\vec{N}^{\,*}\,\mathrm{d}V_{\mathrm{S}^2} = K\,\mathrm{d}V_S$. To prove this, let $\vec{u}, \vec{v} \in \mathrm{T}_{\mathbf{p}}S$ be an orthonormal basis of tangent vector at some point so that $\mathrm{d}V_S(\vec{u}, \vec{v}\,) = 1$ by definition of $\mathrm{d}V$ and $\vec{u} \times \vec{v} = \vec{N}(\mathbf{p})$ by definition of $\vec{N}$.
  - Let us write $-\,\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u}\,) = a\vec{u} + b\vec{v}$ and $-\,\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}\,) = c\vec{u} + d\vec{v}$. Then, by definition, $K(\mathbf{p}) = \det(\mathcal{W}_{\mathbf{p}}) = \det(-\,\mathrm{d}\vec{N}_{\mathbf{p}}) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

– On the other hand,

$$\begin{aligned}(\vec{N}^{*}\,\mathrm{d}V_{\mathrm{S}^2})(\vec{u},\vec{v}) &= \mathrm{d}V_{\mathrm{S}^2}(\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u}),\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}))\\ &= \langle\,\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{u})\times\mathrm{d}\vec{N}_{\mathbf{p}}(\vec{v}),\vec{N}(\mathbf{p})\rangle\\ &= \langle(a\vec{u}+b\vec{v})\times(c\vec{u}+d\vec{v}),\vec{N}(\mathbf{p})\rangle\\ &= \langle(ad-bc)\vec{N}(\mathbf{p}),\vec{N}(\mathbf{p})\rangle\\ &= ad-bc\\ &= K(\mathbf{p})\,\mathrm{d}V_{\mathrm{S}}(\vec{u},\vec{v})\end{aligned}$$

as desired.

- Normal sections
  - Recall that we defined the normal curvature $\mathrm{II}_{\mathbf{p}}(\vec{v})$ of an oriented surface S at a point $\mathbf{p}$ in the direction $\vec{v}$ to be the normal curvature $\langle\vec{N}(\mathbf{p}),\gamma''(t_0)\rangle$ of *any* unit-speed curve γ on S with $\gamma'(t_0)=\vec{v}$.
  - However, one can also define them in terms of certain canonically defined curves called *normal sections*.
  - We consider the plane P through $\mathbf{p}$ parallel to $\vec{N}(\mathbf{p})$ and $\vec{v}$, i.e., with normal vector $\vec{n}=\vec{N}(\mathbf{p})\times\vec{v}$.
  - We now intersect P with S, and we claim that there is some small open set $U\subset\mathbb{R}^3$ containing $\mathbf{p}$ such that the intersection $S\cap P\cap U$ is a smooth curve, i.e., a smooth 1-manifold.
  - Indeed, P is the zero-set of the function $F(\vec{x})=\langle\vec{x}-\mathbf{p},\vec{n}\rangle$ with derivative $\mathrm{d}F(\mathbf{p})=\vec{n}^{\top}$ (i.e., with gradient $\nabla F(\mathbf{p})=\vec{n}$). And since S it is a 2-manifold, there is some neighbourhood W of $\mathbf{p}$ and smooth function $G\colon W\to\mathbb{R}$ with regular value 0 such that $W\cap S=G^{-1}(0)$; moreover, we have that $\nabla G(\mathbf{p})\parallel\vec{N}(\mathbf{p})$, i.e., $\mathrm{d}G(\mathbf{p})=\vec{u}^{\perp}$ for some $\vec{u}\parallel\vec{N}(\mathbf{p})$.
  - It follows that the function $H\colon U\to\mathbb{R}^2$ given by $H(\vec{x})=(F(\vec{x}),G(\vec{x}))$ satisfies $H^{-1}(0)=S\cap P\cap W$, and $\mathrm{d}H(\mathbf{p})$ has linearly independent (in fact, orthogonal) rows $\vec{n}^{\top}$ and $\vec{u}^{\top}$, and hence has $\mathbf{p}$ as a regular point.
  - It then follows that 0 is a regular value of the restriction of H to some small neighbourhood $U\subset W$ of $\mathbf{p}$, and hence by the inverse function theorem, $C=H^{-1}(0)\cap U=P\cap S\cap U$ is a 1-manifold as claimed.
  - We can thus find a parametrization C near $\mathbf{p}$ by a unit-speed curve $\gamma\colon(\varepsilon,\varepsilon)\to\mathbb{R}^3$ with $\gamma(0)=\mathbf{p}$.
  - The smooth curve γ is called the *normal section* of S at $\mathbf{p}$.
  - By definition, it lies entirely inside the plane P, and hence γ'' lies inside P as well. Since $\gamma''(0)\perp\gamma'(0)$, it follows that $\gamma''\parallel\vec{N}(\mathbf{p})$, and hence that

$$\kappa(0)=|\gamma''(0)|=\langle\gamma''(0),\vec{N}(\mathbf{p})\rangle=|\mathrm{II}_{\mathbf{p}}(\vec{v})|.$$

  - We conclude that *the absolute value of the normal curvature* $\mathrm{II}_{\mathbf{p}}(\vec{v})$ *is the curvature at* $\mathbf{p}$ *of the normal section in the direction* $\vec{v}$.
- Geodesics
  - Let γ be a unit-speed curve on an oriented surface S with unit normal vector field $\vec{N}$.
  - We can decompose $\vec{a}=\gamma''$ as $\vec{a}=\vec{a}^{\parallel}+\vec{a}^{\perp}$ with respect to $\vec{N}$.

- We have already identified the second term in terms of the normal curvature $\vec{a}^{\perp} = \kappa_{\mathrm{n}}\vec{N}$.
- Since γ is unit-speed, $\vec{a}$ is orthogonal to both $\vec{v}$ and $\vec{N}$, and hence is parallel to $R_{90}\vec{v} := \vec{N} \times \vec{v}$, the "90 degree counter-clockwise rotation of γ' with respect to the chosen orientation".
- We now set $\kappa_{\mathrm{g}} = \langle \gamma'', R_{90}\vec{v} \rangle$, so that we have $\vec{a}^{\parallel} = \kappa_{\mathrm{g}} R_{90}(\vec{v})$ and hence
$$\vec{a} = \kappa_{\mathrm{n}} \cdot \vec{N} + \kappa_{\mathrm{g}} \cdot R_{90}(\vec{v}).$$
- The quantity κ_{g} is called the *geodesic curvature* of γ, and measure the bending of γ *relative to* the surface.
- If S is a plane, then κ_{g} is just the signed curvature.
- We see that $\kappa_{\mathrm{g}} = 0$ if and only if $\gamma'' \parallel \vec{N}$; note that whether this holds is independent of orientation.
- Thus, in general, we define a *geodesic* on S to be a regular curve γ such that $\gamma''(t)$ is orthogonal to S for all t.
- Hence, if γ is unit-speed, it is a geodesic if and only if $\kappa_{\mathrm{g}} = 0$.
- In general, we have: *every geodesic γ has constant speed.* This follows immediately from the fact that $\gamma' \perp \gamma''$.
- On a plane, the geodesics are precisely the lines with constant-speed parametrization.
- We now state two fundamental facts about geodesics without proof.
- Proposition 5.3: for each $\mathbf{p} \in S$ and each $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$, there exists a geodesic $\gamma_{\vec{v}} : (a, b) \to S$ (where $0 \in (a, b)$) with $\gamma_{\vec{v}}(0) = \mathbf{p}$ and $\gamma_{\vec{v}}'(0) = \vec{v}$.
  Moreover, it is unique in the sense that for any other geodesic $\hat{\gamma}_{\vec{v}} : (\hat{a}, \hat{b}) \to S$ with these properties, $\gamma_{\vec{v}}$ and $\hat{\gamma}_{\vec{v}}$ agree on their common domain $(a, b) \cap (\hat{a}, \hat{b})$.
- Note that, while we can sometimes choose the domain (a, b) to be all of $\mathbb{R}$ (for example, when S is a plane), this isn't always the case.
  For example, if S is the puncture xy-plane $S = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } (x, y) \neq (0, 0)\}$, and we take $\mathbf{p} = (-1, 0, 0)$ and $\vec{v} = (1, 0, 0)$, then the maximal domain of $\gamma_{\vec{v}}$ is $(-\infty, 1)$.
- This proposition is important because it allows us identify *all* the geodesics on a given surface.
- For example, *every geodesic on* S^2 *is (a part of) a great circle* $\gamma(t) = \cos(at)\vec{u} + \sin(at)\vec{v}$, where $\vec{u}, \vec{v} \in \mathrm{S}^2$ are orthogonal.
  Indeed, the great circles are clearly geodesic (since $\gamma'(t) = -a\sin(at)\vec{u} + a\cos(at)\vec{v} \perp \gamma(t) = \vec{N}(\gamma(t))$), hence for any $\mathbf{p} \in \mathrm{S}^2$ and $\vec{v} \in \mathrm{T}_{\mathbf{p}}S$, the unique geodesic $\gamma_{\vec{v}}$ through $\mathbf{p}$ with $\gamma_{\vec{v}}'(0) = \vec{v}$ must be the great circle $\gamma_{\vec{v}}(t) = \cos(at)\mathbf{p} + \sin(at)\vec{v}$.
- By a similar argument, we can see that *every geodesic on the cylinder* $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ *is a helix* $\gamma(t) = (\cos t, \sin t, ct)$.
- Next, Corollary 5.23: a curve γ is a geodesic if and only if it is *locally length-minimizing.* This means that for any t_0 in the domain of γ, there is some $\varepsilon > 0$ such that for any $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$, if we set $\mathbf{p} = \gamma(t_1)$ and $\mathbf{q} = \gamma(t_2)$, then $\gamma|_{[t_1,t_2]}$ is the shortest path on S from $\mathbf{p}$ to $\mathbf{q}$, i.e., any other curve $\alpha\colon [a, b] \to S$ from $\mathbf{p}$ to $\mathbf{q}$ on S has greater arc-length than $\gamma|_{[t_1,t_2]}$:
$$\int_a^b |\alpha'(t)|\,\mathrm{d}t \geq \int_{t_1}^{t_2} |\gamma(t)|\,\mathrm{d}t = |\gamma'|(t_2 - t_1)$$
  where for the last equation, we are using that γ has constant speed.

  - In particular this tells us immediately that (Corollary 5.24): geodesics are intrinsic.
  - This gives us a second proof that the helices are the geodesics on the cylinder, since we know what the geodesics in the plane are.
- Gauss-Bonnet
  - We finish our exploration of surfaces (though there is much more we are not covering – just look at some of the other sections in Tapp's book!) with the spectacular *Gauss-Bonnet* theorem (which we will discuss, but not prove).

    The theorem is analogue of Hopf's Umlaufsatz, saying that the *total signed curvature* $\int_a^b \kappa_{\mathrm{s}}$ of a simple closed curve is $\pm 2\pi$.

    The Gauss-Bonnet theorem gives a formula for the *total Gaussian curvature* $\int_S K\,\mathrm{d}V$ of a compact surface S.

    These are both *global* theorems, in that they something about the given curve or surface as a whole, rather than describing the behaviour in a small or infinitesimal neighbourhood of each point (for example, giving formulas relating speed, curvature, and so on).
  - To state the formula, we first need to introduce some auxiliary notions.
  - A *polygonal region* on a surface S (possibly with boundary) is the a subset $R \subset S$ which is the image under some surface patch $\sigma\colon U \xrightarrow{\sim} V \subset S$ of a piecewise smooth curve in U together with its interior.
  - A *triangle* on S is a polygonal region with three vertices. The three smooth segments of the boundary are called its *edges*.
  - A *triangulation* of S is a set $\mathcal{T}$ of triangles whose union is S, and such that for any two $T_1, T_2 \in \mathcal{T}$, the intersection $T_1 \cap T_2$ is either an edge of both T_1, T_2 or a vertex of both T_2, T_1.
  - It is a (non-trivial) fact that every surface with boundary admits a finite triangulation.
  - If $\mathcal{T}$ is a finite triangulation, its *Euler characteristic* of $\mathcal{T}$ is the alternating sum $\chi = V - E + F$, where V, E, F are the number of vertices, edges, and triangles appearing in the triangulation.
  - We also call this *the Euler characteristic of* S because – as we will see right now – it is independent of the triangulation (though this is far from obvious).
  - Namely, the *Gauss-Bonnet theorem* says that for a compact surface S (without boundary!), we have:
$$\int_S K = 2\pi\chi(S).$$
  - Since the left side is independent of the triangulation, this shows that the right side is too.
  - On the other hand, the right side is clearly invariant under *diffeomorphism* (since you can just take the image of any triangulation under the diffeomorphism), hence this shows the remarkable fact that the *total Gaussian curvature is invariant under diffeomorphism* (and not just isometry)!

Math 435: Lecture 42

April 26, 2024

Reference: Tapp, pp. Chapters 4 and 5

Topics:

- Examples of Gauss-Bonnet
  - For the unit sphere S^2, we have constant curvature $K = 1$, and area 4π, hence we find that $\chi(S^2) = 2$.
  - We can also compute it directly: for example, if we triangulate S^2 as a tetrahedron, we obtain $\chi = 4 - 6 + 4 = 2$, or if as an octahedron, we obtain $\chi = 8 - 12 + 6 = 2$.
  - A useful fact is that you can actually compute the Euler characteristic using *any* partition into polygons, not just triangles. The reason is that you can always subdivide a polygon into triangles, and it won't change the Euler characteristic, since each time we connect two corners, we are adding both a face and an edge.
  - Hence, we can also compute $\chi(S^2)$ with a cube $6 - 12 + 8 = 2$.
  - Next, for the *torus*, it is harder to compute the total Gaussian curvature directly. But we can triangulate (or "polygonate") it using a single face, two edges, and one vertex, hence $\xi = 1 - 2 + 1 = 0$. Thus the *total Gaussian curvature of a torus is 0.*
  - There is a second way to see this, which is by attaching a *handle* to the sphere.
  - That is, we consider the cylinder $\{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } z \in [0, 1]\}$. It has a polygonation with 1 face, 3 edges, and 2 vertices (hence $\chi = 0$).
  - We can now start with a polygonated surface, remove two faces, and glue on a cylinder along the two resulting boundaries; altogether, we will have removed two faces, and added one face and one edge, hence decreased χ by 2.
  - If we add a handle to S^2, we obtain a torus, so we see again that χ of the torus is 2.
  - In general, the *genus* $g(S)$ of a compact surface $S \subset \mathbb{R}^3$ is its number of handles (equivalently, the number of "cuts" that need to be made in order to turn it into a sphere); hence a surface has genus 0, a torus has genus 1, and if we attach another handle, we get a surface of genus 2 (a kind of "double torus").
  - Since $\chi(S^2) = 2$, we see in general that the Euler characteristic of a genus g surface is
    $$\chi = 2 - 2g.$$
  - Note that besides the two special cases $g = 0$ (the sphere) and $g = 1$ (the sphere) the *total Gaussian curvature is always negative.*
- Gauss-Bonnet with boundary
  - The proof of Gauss-Bonnet involves introducing two generalizations of it, which are interesting in their own right.
  - The first concerns surfaces with boundary, and says that if S is a surface with boundary, then
    $$\int_S K + \int_{\partial S} \kappa_g = 2\pi\chi(S). \tag{4}$$
    (Note that κ_g in principle depends on an orientation of S, but switching the orientation changes both the direction of the induced orientation on the boundary curve, and the sign of κ_g, so these will cancel out.)
  - Note that if $\partial S = \emptyset$, this recovers the old formula.

  - In particular, this shows that the Euler characteristic of a surface *with boundary* is also independent of the triangulation (which we didn't know yet).
  - The second variant is about a *polygonal region* R on a surface S – i.e., it is the case where we have not only boundaries, but corners. In this case, the formula is:

$$\int_R K + \int_{\partial R} \kappa_{\mathrm{g}} + \sum_i \alpha_i = 2\pi \tag{5}$$

    where the sum $\sum_i \alpha_i$ is over the external angles at the corners of the polygon. Note that we might also write the right side as $2\pi\chi(R)$ since the Euler characteristic of a disk is $1 - 1 + 1 = 1$.
  - Compare this with the Umlaufsatz: $\int_a^b \kappa_{\mathrm{s}}(t) + \sum_{i=1}^n \alpha_i = 2\pi$.
  - (As stated, this version isn't quite a *generalization* of (4), since a whole surface will not in general be a polygonal region, but one can formulate a more general version of the theorem which really is a generalization of (4), see Tapp Theorem 6.8.)
- Geodesic polygons
  - Though the polygonal version (5) of Gauss-Bonnet will mainly be a tool for us to prove the version (4) for a whole surface, it is actually very interesting in its own right, especially in the case of *geodesic polygons*, i.e., polygons whose edges are geodesics, so that the term κ_{g} vanishes.
  - In the plane, where $K = 0$, it then just reduces to the formula $\sum_i \alpha_i = 2\pi$, or the equivalent version with internal angles $\sum \beta_i = (n-2)\pi$.
  - Another interesting example to consider is that of *spherical geometry*, i.e., the study of figures formed out of great circles on the sphere.
  - Here, we see, for example, that the usual angle sum of π for a triangle is increased (the triangles are "fatter"), and moreover the defect $\int_R K$ is proportional to the *area* of the triangle.
    (In particular, *very small* triangles have angle sum nearly π, as on a small scale, any surface is "nearly flat".)
  - There is a third famous case – called *hyperbolic geometry* – which is that of a surface with constant negative rather than positive Gaussian curvature, $K = -1$.
  - It is difficult to come across such surfaces in $\mathbb{R}^3$, but there is a famous "abstract" surface with this property called the *hyperbolic plane*, which we do not have time to explain now.
  - In any case, here, the triangles are instead *thinner*, and the defect again is proportional to the area of the triangle.
  - The dutch artist M.C. Escher made some famous and beautiful artistic renditions of the hyperbolic plane, which you should look at.
- Sketch of the proof
  - Proving the Gauss-Bonnet theorem has two steps: first prove (5), and then use that to prove (4).
  - The proof of (5) is the more difficult but (somewhat) less interesting part. There are two basic ideas involved: (i) use Stokes' theorem to relate the integral $\int_R K$ over the interior to the integral $\int \kappa_{\mathrm{g}}$ over the boundary, and (ii) use a variant of the argument from the Umlaufsatz (including the part about smoothing the corners) to relate the integral $\int \kappa_{\mathrm{g}}$ to a "total change of angle", which again comes out to be 2π.

– The fun part of the proof is deducing the Gauss-Bonnet theorem (4) for a whole surface, assuming the version (5) about polygonal regions.
– Choose a finite triangulation $T_1, \dots T_F$ of the given surface(-with-boundary) S.
– By assumption the equation (5) holds with $R = T_i$ for each i.
– Let us sum both sides of this equation over all i. On the right-hand side, we just get $2\pi F$.
– The first term on the left-hand side simply adds up to $\int_S K$.
– The sum of the terms $\int_{\partial T_i} \kappa_{\mathrm{g}}$ give the sum over all the edges e of all the triangles of $\int_e \kappa_{\mathrm{g}}$. However, each *interior* edge (i.e., edge which is not on a boundary) is counted twice with opposite signs (because it has the opposite induced orientation from the two adjacent triangles), hence these cancel out. We are thus left with $\int_{\partial S} \kappa_{\mathrm{g}}$.
– Finally, the terms $\sum_i \alpha_i$ add up to the sum of all the external angles of all the vertices of all the triangles.
– Recall that the external angle α_i is $\pi - \beta_i$, where β_i is the internal angle.
– Hence, we can write this as $A\pi - \sum \beta_i$, where A is the total number of angles, and the second term is the sum of all the *internal* angles (at all vertices).
– Now the sum of all the internal angles at an *interior* vertex is 2π, whereas at an *exterior* vertex (i.e., one lying on ∂S), it is just π. Hence $\sum \beta_i = 2\pi V_{\mathrm{int}} + \pi V_{\mathrm{ext}}$.
– On the other hand, the number angles at a given interior vertex v is simply the number of edges E_v emanating from V, whereas at an exterior vertex it is $E_v - 1$. Hence, since each edge occurs at two vertices, summing them all up gives $A = 2E - V_{\mathrm{ext}}$.
– Hence, we have $A\pi - \sum \beta_i = (2E - V_{\mathrm{ext}})\pi - (2\pi V_{\mathrm{int}} + V_{\mathrm{ext}}) = 2\pi(E - V)$.
– Adding everything up, we thus have

$$\int_S K + \int_{\partial S} \kappa_{\mathrm{g}} + 2\pi(E - V) = 2\pi F,$$

which is the Gauss-Bonnet formula.